A Level Pure Mathematics
Practice Test 5: Trigonometry

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Radian Measure and Basic Functions

1. Convert these angles from degrees to radians (leave answers in terms of π):
   (a) $22.5°$
   (b) $67.5°$
   (c) $112.5°$
   (d) $157.5°$
   (e) $202.5°$
   (f) $337.5°$
2. Convert these angles from radians to degrees:
   (a) $\frac{\pi}{16}$
   (b) $\frac{3\pi}{16}$
   (c) $\frac{9\pi}{8}$
   (d) $\frac{13\pi}{4}$
   (e) $\frac{14\pi}{3}$
   (f) $\frac{19\pi}{6}$
3. Find the exact values of these trigonometric ratios (without calculator):
   (a) $\sin(\frac{7\pi}{2})$, $\cos(\frac{7\pi}{2})$, $\tan(\frac{7\pi}{2})$
   (b) $\sin(\frac{9\pi}{2})$, $\cos(\frac{9\pi}{2})$, $\tan(\frac{9\pi}{2})$
   (c) $\sin(-2\pi)$, $\cos(-2\pi)$, $\tan(-2\pi)$
   (d) $\sin(-\frac{5\pi}{2})$, $\cos(-\frac{5\pi}{2})$, $\tan(-\frac{5\pi}{2})$
4. A circle has radius 18 cm. Find:
   (a) The arc length subtended by an angle of $\frac{11\pi}{12}$ radians
   (b) The area of the sector with angle $\frac{8\pi}{15}$ radians
   (c) The angle (in radians) that subtends an arc of length 42 cm
   (d) The radius of a circle where an angle of $\frac{7\pi}{12}$ radians subtends an arc of length 35 cm
5. Find the exact values:

(a) $\sin \frac{10\pi}{3}$
(b) $\cos \frac{11\pi}{4}$
(c) $\tan \frac{7\pi}{3}$
(d) $\sin \frac{14\pi}{3}$
(e) $\cos \frac{15\pi}{4}$
(f) $\tan \frac{17\pi}{6}$

Section B: Graphs of Trigonometric Functions

6. For the function $f(x) = \frac{3}{2} \sin x$:
   (a) State the domain and range
   (b) Find the period
   (c) Identify the zeros in the interval $[0, 2\pi]$
   (d) Find the maximum and minimum values and where they occur
   (e) Sketch the graph for $x \in [-2\pi, 2\pi]$

7. For the function $g(x) = \cos 2.5x$:
   (a) State the domain and range
   (b) Find the period
   (c) Identify the zeros in the interval $[0, 2\pi]$
   (d) Find the maximum and minimum values and where they occur
   (e) Sketch the graph for $x \in [0, 2\pi]$

8. For the function $h(x) = \tan \frac{2x}{3}$:
   (a) State the domain and range
   (b) Find the period
   (c) Identify the asymptotes in the interval $[0, 3\pi]$
   (d) Find the zeros in the interval $[0, \frac{3\pi}{2}]$
   (e) Sketch the graph for $x \in [0, 3\pi]$

9. Sketch the graphs of these transformed functions for $x \in [0, 4\pi]$:
   (a) $y = \frac{3}{2} \sin x$
   (b) $y = \cos \frac{2x}{3}$
   (c) $y = \sin(x - \frac{\pi}{2})$
   (d) $y = \cos x - \frac{3}{2}$
   (e) $y = -\frac{1}{2} \sin x$
   (f) $y = \tan(x + \frac{\pi}{8})$

10. For the function $y = 6\sin(\frac{3x}{2} - \frac{\pi}{4}) - 4$:
   (a) Identify the amplitude
   (b) Find the period
   (c) Determine the phase shift
   (d) Find the vertical shift
   (e) State the range
   (f) Sketch the graph for $x \in [0, \frac{8\pi}{3}]$

Section C: Fundamental Trigonometric Identities

11. Use the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ to find:
    (a) $\cos\theta$ if $\sin\theta = \frac{11}{61}$ and θ is acute
    (b) $\sin\theta$ if $\cos\theta = -\frac{9}{40}$ and θ is in the third quadrant
    (c) $\tan\theta$ if $\sin\theta = \frac{21}{29}$ and $\cos\theta > 0$
    (d) $\cos\theta$ if $\tan\theta = -\frac{33}{56}$ and $\sin\theta < 0$
12. Prove these cofunction identities:
    (a) $\sin(\frac{\pi}{2} - \theta) = \cos\theta$
    (b) $\cos(\frac{\pi}{2} - \theta) = \sin\theta$
    (c) $\tan(\frac{\pi}{2} - \theta) = \cot\theta$
    (d) $\sec(\frac{\pi}{2} - \theta) = \csc\theta$
13. Simplify these expressions:
    (a) $\cos^2\theta(\sec^2\theta - 1)$
    (b) $\frac{\csc\theta}{\cot\theta} + \frac{\sec\theta}{\tan\theta}$
    (c) $(\sec\theta - \cos\theta)^2$
    (d) $\frac{\csc^2\theta - 1}{\cot\theta}$
14. Express in terms of $\csc\theta$ only:
    (a) $\sin^2\theta$
    (b) $\cos^2\theta$
    (c) $\cot^2\theta$
    (d) $\sin^2\theta + \cos^2\theta\sec^2\theta$
15. Prove that:
    (a) $\frac{\cot^2\theta}{\csc\theta + 1} = \csc\theta - 1$
    (b) $\csc^4\theta - \cot^4\theta = 1 + 2\cot^2\theta$
    (c) $\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \frac{1 + \tan\theta}{1 - \tan\theta}$
    (d) $(1 + \cos\theta)(1 - \cos\theta) = \sin^2\theta$

Section D: Addition and Double Angle Formulas

16. Use the addition formulas to find the exact values:
    (a) $\sin 195°$ (using $\sin(150° + 45°)$)
    (b) $\cos 285°$ (using $\cos(240° + 45°)$)
    (c) $\tan 255°$ (using $\tan(210° + 45°)$)
    (d) $\sin\frac{17\pi}{12}$ (using $\sin(\frac{5\pi}{4} + \frac{\pi}{6})$)
17. Given $\sin A = \frac{21}{29}$ with A acute and $\cos B = \frac{24}{25}$ with B acute:
    (a) Find $\cos A$ and $\sin B$
    (b) Calculate $\cos(A + B)$
    (c) Calculate $\sin(A - B)$
    (d) Find $\tan(B - A)$

18. Use double angle formulas to find:
    (a) $\sin 2\theta$ if $\cos\theta = \frac{21}{29}$ and θ is acute
    (b) $\cos 2\theta$ if $\sin\theta = \frac{33}{65}$ and θ is acute
    (c) $\tan 2\theta$ if $\tan\theta = \frac{11}{60}$
    (d) $\cos 2\theta$ if $\sin\theta = -\frac{16}{65}$ and θ is in the third quadrant

19. Derive these triple angle formulas:
    (a) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$
    (b) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$
    (c) $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$
    (d) $\cot 3\theta = \frac{\cot^3\theta - 3\cot\theta}{3\cot^2\theta - 1}$

20. Express in multiple angle form:
    (a) $8\sin^3\theta$ in terms of $\sin\theta$ and $\sin 3\theta$
    (b) $8\cos^3\theta$ in terms of $\cos\theta$ and $\cos 3\theta$
    (c) $\sin^3\theta\cos^3\theta$ in terms of $\sin 6\theta$
    (d) $16\sin^4\theta\cos^4\theta$ in terms of $\cos 8\theta$

Section E: Solving Trigonometric Equations

21. Solve these equations for $0 \le x \le 2\pi$:
    (a) $\sin x = \frac{\sqrt{2}}{2}$
    (b) $\cos x = -\frac{1}{\sqrt{2}}$
    (c) $\tan x = -\frac{1}{\sqrt{3}}$
    (d) $\sin x = -\frac{1}{\sqrt{2}}$

22. Solve these equations for $0° \le x \le 360°$:
    (a) $4\sin x - 3 = 0$
    (b) $2\cos x + \sqrt{3} = 0$
    (c) $\sqrt{3}\tan x + 1 = 0$
    (d) $2\sin^2 x = \sqrt{3}\sin x$

23. Solve these equations for $0 \le x \le 2\pi$:
    (a) $\sin 4x = -\frac{\sqrt{3}}{2}$
    (b) $\cos\frac{x}{3} = \frac{1}{2}$
    (c) $\tan 6x = -1$
    (d) $\sin(x + \frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

24. Solve these quadratic trigonometric equations for $0 \le x \le 2\pi$:
    (a) $4\sin^2 x - 3\sin x - 1 = 0$
    (b) $3\cos^2 x + \cos x - 2 = 0$
    (c) $2\tan^2 x + 3\tan x + 1 = 0$
    (d) $6\sin^2 x - 7\sin x + 2 = 0$

25. Solve these equations involving multiple angles for $0 \leq x \leq 2\pi$:
    (a) $\sec x = \csc x$
    (b) $\cos 2x = \sin x$
    (c) $\sin 3x = \cos x$
    (d) $\tan 6x = \tan 2x$

Section F: Advanced Trigonometric Identities

26. Prove these Mollweide formulas for triangle ABC:
    (a) $\frac{a+b}{c} = \frac{\cos(\frac{A-B}{2})}{\sin(\frac{C}{2})}$
    (b) $\frac{a-b}{c} = \frac{\sin(\frac{A-B}{2})}{\cos(\frac{C}{2})}$
    (c) $\frac{b+c}{a} = \frac{\cos(\frac{B-C}{2})}{\sin(\frac{A}{2})}$
    (d) $\frac{b-c}{a} = \frac{\sin(\frac{B-C}{2})}{\cos(\frac{A}{2})}$

27. Use prosthaphaeresis formulas to compute:
    (a) $\sin 9x + \sin 3x$
    (b) $\cos 11x - \cos 5x$
    (c) $\sin 105° + \sin 75°$
    (d) $\cos 127.5° - \cos 52.5°$

28. Prove these Euler formulas using complex exponentials:
    (a) $\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$
    (b) $\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$
    (c) $\tan \theta = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}$
    (d) $e^{i\theta} = \cos \theta + i \sin \theta$

29. Express using Euler's formula:
    (a) $\sin^5 \theta$ as a sum of sines
    (b) $\cos^5 \theta$ as a sum of cosines
    (c) $\sin \theta \cos^4 \theta$ as a sum
    (d) $\cos^3 \theta \sin^2 \theta$ as a sum

30. Prove the sextuple angle formulas:
    (a) $\sin 6\theta = 32 \sin^6 \theta - 48 \sin^4 \theta + 18 \sin^2 \theta - 1$ (when $\cos \theta = 0$)
    (b) $\cos 6\theta = 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1$
    (c) $\tan 6\theta = \frac{6 \tan \theta - 20 \tan^3 \theta + 6 \tan^5 \theta}{1 - 15 \tan^2 \theta + 15 \tan^4 \theta - \tan^6 \theta}$

Section G: Complex Trigonometric Problems

31. Solve these equations for $0 \leq x < 2\pi$:
    (a) $2\sin x + \cos x = 1$
    (b) $\sin x - \sin 3x = 0$
    (c) $\cos x + \cos 5x + \cos 7x = 0$
    (d) $\csc x - \csc 2x = 0$

32. Prove these elegant identities:
    (a) $\frac{\sin 6\theta}{\sin\theta} + \frac{\cos 6\theta}{\cos\theta} = 2\cos 5\theta$
    (b) $\cos^2\theta + \cos^2(\theta + \frac{\pi}{3}) + \cos^2(\theta - \frac{\pi}{3}) = \frac{3}{2}$
    (c) $\sin^8\theta + \cos^8\theta = 1 - 2\sin^2\theta\cos^2\theta + 2\sin^4\theta\cos^4\theta$
    (d) $\cos A\cos B\cos C = \frac{1}{4}[\cos(A+B+C) + \cos(A+B-C) + \cos(A-B+C) + \cos(-A+B+C)]$

33. Find the general solution to these equations:
    (a) $\cos x = \frac{5}{7}$
    (b) $\sin 6x = -0.7$
    (c) $\tan\frac{2x}{3} = 5$
    (d) $\cos(6x + \frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$

34. Express these in the form $R\sin(x + \alpha)$ or $R\cos(x + \alpha)$:
    (a) $24\sin x + 7\cos x$
    (b) $15\sin x - 8\cos x$
    (c) $5\sin x + 5\cos x$
    (d) $8\cos x - 8\sqrt{3}\sin x$

35. Find the range of these functions:
    (a) $f(x) = 24\sin x + 7\cos x$
    (b) $g(x) = 15\sin 5x - 8\cos 5x + 12$
    (c) $h(x) = \sin^2 x - 6\cos x$
    (d) $k(x) = 10\sin x\cos x + 7$

Section H: Applications of Trigonometry

36. A pendulum bob oscillates with angular displacement $\theta = 0.4\cos(6t + \frac{\pi}{3})$ radians, where t is time in seconds.
    (a) Find the maximum angular displacement
    (b) Determine the period of oscillation
    (c) Find the phase shift
    (d) Calculate the angular displacement when $t = 0$
    (e) Find when the bob first reaches maximum positive displacement

37. The water level in a tidal pool is modeled by $L(t) = 1.5\cos(\frac{\pi t}{8} + \frac{\pi}{4}) + 2.5$ meters, where t is hours after midnight.
    (a) Find the maximum and minimum water levels

(b) Determine the period of the tidal cycle
(c) Find the water level at 5 AM
(d) Calculate when the water level is at its maximum
(e) Find when the water level is exactly 3.5 meters

38. A radio signal is modulated with $V = 8\cos(10^4\pi t - \frac{\pi}{6})$ volts.

(a) Find the maximum voltage amplitude
(b) Determine the frequency of the signal
(c) Calculate the voltage when $t = 5 \times 10^{-5}$ seconds
(d) Find when the voltage first equals 4 volts
(e) Determine the phase angle in degrees

39. A robotic arm rotates with angular position $\alpha = \frac{\pi}{3}\sin(4t - \frac{\pi}{6})$ radians, where t is time in seconds.

(a) Find the maximum angular displacement
(b) Determine the period of rotation
(c) Calculate the angular position at $t = 0$
(d) Find when the arm first reaches minimum displacement
(e) Determine the angular acceleration at $t = \frac{\pi}{8}$ seconds

40. Two acoustic waves with equations $A_1 = 6\sin 7x$ and $A_2 = 8\cos 7x$ combine.

(a) Find the equation of the combined wave
(b) Express the result in the form $R\sin(7x + \alpha)$
(c) Determine the amplitude of the combined wave
(d) Find the phase relationship between the original waves
(e) Calculate the beat frequency if these represent sound waves

Section I: Advanced Problem Solving

41. In triangle GHI, $g = 17$, $h = 24$, and $\angle I = 105°$.

(a) Use the cosine rule to find side i
(b) Use the sine rule to find $\angle G$
(c) Calculate the area of the triangle
(d) Find the radius of the nine-point circle
(e) Determine the length of the symmedian from I

42. Prove that in any triangle GHI:

(a) $\frac{g}{\sin G} = \frac{h}{\sin H} = \frac{i}{\sin I} = 2R$ (sine rule)
(b) $g^2 = h^2 + i^2 - 2hi\cos G$ (cosine rule)
(c) $\sin G + \sin H + \sin I = 4\cos\frac{G}{2}\cos\frac{H}{2}\cos\frac{I}{2}$
(d) The sum of any two sides is to the third side as the sum of sines of opposite angles is to the sine of the third angle

43. A regular dodecagon is inscribed in a circle of radius r.

(a) Find the central angle for each sector
(b) Calculate the side length of the dodecagon

(c) Find the area of the dodecagon
(d) Determine the area of the 12 triangular sectors
(e) Calculate the ratio of the dodecagon's area to the inscribed hexagon's area

44. The function $k(x) = u \sin 4x + v \cos 4x$ has maximum value 40 and minimum value -40.

(a) Express $k(x)$ in the form $R\cos(4x - \epsilon)$
(b) Find the relationship between u and v
(c) If $k(\frac{\pi}{16}) = 32$, find the values of u and v
(d) Solve $k(x) = 24$ for $0 \leq x \leq \frac{\pi}{2}$
(e) Find the inflection points of $k(x)$ in the interval $[0, \frac{\pi}{2}]$

45. Consider the identity for regular polygons: For a regular n-gon, $\sum_{k=1}^{n-1} \sin \frac{2\pi k}{n} = 0$.

(a) Verify this identity for $n = 6$ (regular hexagon)
(b) Prove this identity using complex numbers
(c) Find $\sum_{k=1}^{11} \cos \frac{2\pi k}{12}$ for a regular dodecagon
(d) Use this to find the exact value of $\cos \frac{\pi}{12}$
(e) Apply these results to compute the area of a regular 24-gon

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

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