A Level Pure Mathematics
Practice Test 1: Differential Equations

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Introduction to Differential Equations

1. Define the following terms and give examples:
   (a) Differential equation
   (b) Order of a differential equation
   (c) Degree of a differential equation
   (d) General solution
   (e) Particular solution
   (f) Initial condition

2. Classify these differential equations by order and degree:
   (a) $\frac{dy}{dx} = 3x + 2$
   (b) $\frac{d^2y}{dx^2} + 4y = 0$
   (c) $\left(\frac{dy}{dx}\right)^2 + y = x$
   (d) $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + y = e^x$
   (e) $x\frac{dy}{dx} + y^2 = 1$
   (f) $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^3 + y = 0$

3. Verify that the given functions are solutions to the differential equations:
   (a) $y = 3e^{2x}$ satisfies $\frac{dy}{dx} = 2y$
   (b) $y = A\cos x + B\sin x$ satisfies $\frac{d^2y}{dx^2} + y = 0$
   (c) $y = x^2 + 2x + C$ satisfies $\frac{dy}{dx} = 2x + 2$
   (d) $y = Ce^{-x^2/2}$ satisfies $\frac{dy}{dx} + xy = 0$

4. Form differential equations from the given families of curves:
   (a) $y = Ae^{3x}$ (where A is an arbitrary constant)
   (b) $y = A\cos(2x) + B\sin(2x)$ (where A and B are arbitrary constants)
   (c) $x^2 + y^2 = r^2$ (where r is an arbitrary constant)

   (d) $y = Cx^2$ (where C is an arbitrary constant)

5. Explain the difference between:

   (a) Ordinary and partial differential equations
   (b) Linear and non-linear differential equations
   (c) Homogeneous and non-homogeneous differential equations
   (d) Separable and non-separable differential equations

Section B: First-Order Differential Equations - Direct Integration

6. Solve these differential equations by direct integration:

   (a) $\frac{dy}{dx} = 6x^2$
   (b) $\frac{dy}{dx} = e^{3x}$
   (c) $\frac{dy}{dx} = \frac{1}{x+1}$
   (d) $\frac{dy}{dx} = \cos(2x)$
   (e) $\frac{dy}{dx} = \frac{2x}{x^2+1}$
   (f) $\frac{dy}{dx} = x^2 + 3x - 2$

7. Solve these differential equations with given initial conditions:

   (a) $\frac{dy}{dx} = 4x^3$, $y(0) = 2$
   (b) $\frac{dy}{dx} = e^x$, $y(0) = 5$
   (c) $\frac{dy}{dx} = \sin x$, $y(\pi) = 1$
   (d) $\frac{dy}{dx} = \frac{1}{x}$, $y(1) = 3$ (for $x > 0$)
   (e) $\frac{dy}{dx} = 2x + 1$, $y(1) = 4$

8. Find the particular solution to each differential equation:

   (a) $\frac{dy}{dx} = 3x^2 - 2x + 1$, passing through $(0, 2)$
   (b) $\frac{dy}{dx} = e^{2x} + 3$, passing through $(0, 1)$
   (c) $\frac{dy}{dx} = \cos x - \sin x$, passing through $(\pi/2, 0)$
   (d) $\frac{dy}{dx} = \frac{1}{x^2+1}$, passing through $(0, \pi)$

9. A particle moves along a straight line such that its velocity is given by $v = \frac{ds}{dt} = 6t^2 - 4t + 1$.

   (a) Find the displacement s as a function of time t
   (b) If $s = 0$ when $t = 0$, find the particular solution
   (c) Find the displacement after 3 seconds
   (d) Find the acceleration as a function of time

10. The gradient of a curve at any point (x, y) is $3x^2 + 2x - 1$.

   (a) Find the general equation of the curve
   (b) Find the particular equation if the curve passes through $(1, 5)$
   (c) Find the coordinates where the gradient is zero
   (d) Sketch the curve showing key features

Section C: Separation of Variables

11. Solve these separable differential equations:
    (a) $\frac{dy}{dx} = xy$
    (b) $\frac{dy}{dx} = \frac{y}{x}$
    (c) $\frac{dy}{dx} = y^2$
    (d) $\frac{dy}{dx} = \frac{x}{y}$
    (e) $\frac{dy}{dx} = e^{x+y}$
    (f) $\frac{dy}{dx} = \frac{x^2}{y^2}$

12. Solve with the given initial conditions:
    (a) $\frac{dy}{dx} = 2xy$, $y(0) = 1$
    (b) $\frac{dy}{dx} = \frac{y}{x}$, $y(1) = 2$ (for $x > 0$)
    (c) $\frac{dy}{dx} = \frac{y^2}{x^2}$, $y(1) = 1$
    (d) $\frac{dy}{dx} = y(1-y)$, $y(0) = 0.5$
    (e) $\frac{dy}{dx} = \frac{x}{\sqrt{y}}$, $y(0) = 1$

13. Find particular solutions to these equations:
    (a) $y\frac{dy}{dx} = x$, passing through $(2, 1)$
    (b) $\frac{dy}{dx} = \frac{y\cos x}{\sin x}$, passing through $(\pi/2, 3)$
    (c) $(1+x^2)\frac{dy}{dx} = y^2$, passing through $(0, 1)$
    (d) $x\frac{dy}{dx} = y\ln y$, passing through $(1, e)$

14. Solve these more complex separable equations:
    (a) $\frac{dy}{dx} = \frac{x(1+y^2)}{y(1+x^2)}$
    (b) $\frac{dy}{dx} = \frac{ye^x}{x^2}$
    (c) $\cos y\frac{dy}{dx} = \sin x$
    (d) $\frac{dy}{dx} = \frac{x^2+1}{y^2+1}$
    (e) $\frac{dy}{dx} = \frac{xy^2}{x^2+1}$

15. Determine which of these equations are separable and solve those that are:
    (a) $\frac{dy}{dx} = x + y$
    (b) $\frac{dy}{dx} = xy + x$
    (c) $\frac{dy}{dx} = x^2 + y^2$
    (d) $\frac{dy}{dx} = \frac{x+y}{x-y}$
    (e) $\frac{dy}{dx} = e^{x-y}$
    (f) $\frac{dy}{dx} = \sin(x+y)$

Section D: Applications of First-Order Differential Equations

16. Population growth models:
    (a) A population grows at a rate proportional to its size. If $P(t)$ represents population at time t and the growth rate is 3% per year, set up and solve the differential equation.
    (b) If the initial population is 1000, find the population after 10 years.
    (c) How long does it take for the population to double?
    (d) What assumptions does this model make about population growth?
17. Radioactive decay:
    (a) The rate of radioactive decay is proportional to the amount present. Set up the differential equation for substance with decay constant k.
    (b) If half-life is 1000 years, find the decay constant k.
    (c) If initially there are 100g, how much remains after 2000 years?
    (d) How long until only 10g remains?
18. Newton's law of cooling:
    (a) A body cools at a rate proportional to the temperature difference with its surroundings. Set up the differential equation.
    (b) A cup of coffee at 90°C is placed in a room at 20°C. After 5 minutes it has cooled to 70°C. Find the temperature after 15 minutes.
    (c) How long does it take to reach 30°C?
    (d) What is the limiting temperature as $t \to \infty$?
19. Mixing problems:
    (a) A tank contains 100L of pure water. Salt water with concentration 0.5 kg/L flows in at 2 L/min, and the mixture flows out at the same rate. Set up and solve for the salt concentration over time.
    (b) Find the concentration after 1 hour.
    (c) What is the limiting concentration as $t \to \infty$?
    (d) Sketch the concentration vs. time graph.
20. Economic models:
    (a) The rate of price increase is proportional to the excess demand. If demand exceeds supply by 100 units and this causes a 2% price increase per month, model this situation.
    (b) If current price is £50, find the price after 6 months.
    (c) An investment grows continuously at 5% per annum. If £1000 is invested, find the value after t years.
    (d) Compare with compound interest calculated annually.

Section E: Linear First-Order Differential Equations

21. Identify which equations are linear and solve using integrating factors:
    (a) $\frac{dy}{dx} + 2y = e^x$
    (b) $\frac{dy}{dx} - 3y = x^2$
    (c) $\frac{dy}{dx} + \frac{y}{x} = x$ (for $x > 0$)

(d) $\frac{dy}{dx} + y \tan x = \sec x$

(e) $x\frac{dy}{dx} + 2y = x^3$

(f) $\frac{dy}{dx} + y^2 = x$ (is this linear?)

22. Solve these linear first-order equations with initial conditions:

(a) $\frac{dy}{dx} + y = 2e^x$, $y(0) = 1$

(b) $\frac{dy}{dx} - 2y = 4x$, $y(0) = 3$

(c) $\frac{dy}{dx} + 3y = 6$, $y(0) = 0$

(d) $\frac{dy}{dx} + \frac{y}{x} = 2x$, $y(1) = 4$ (for $x > 0$)

23. Find the integrating factor and solve:

(a) $\frac{dy}{dx} + y \cos x = \sin x \cos x$

(b) $\frac{dy}{dx} - y \cot x = 2 \cos x$

(c) $(1 + x)\frac{dy}{dx} + y = (1 + x)^2$

(d) $x\frac{dy}{dx} + y = x^2 + 1$ (for $x > 0$)

24. Applications of linear differential equations:

(a) An RC circuit has resistance 10 and capacitance 0.01F. If a 12V battery is connected at $t = 0$, find the current $i(t)$ given $\frac{di}{dt} + \frac{i}{RC} = 0$.

(b) A spring-mass system with damping satisfies $m\frac{dv}{dt} + cv = mg$ where v is velocity. Solve for $v(t)$ if $v(0) = 0$.

(c) A savings account earns 4% interest continuously, but £1000 is withdrawn each year. Model this with a differential equation and solve.

25. Compare different solution methods:

(a) For $\frac{dy}{dx} = 2xy + x$, solve by separation of variables

(b) Solve the same equation using integrating factor method

(c) Verify both solutions are equivalent

(d) Discuss when each method is more appropriate

Section F: Second-Order Differential Equations - Introduction

26. Verify that these functions satisfy the given differential equations:

(a) $y = Ae^{2x} + Be^{-3x}$ satisfies $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$

(b) $y = A\cos(3x) + B\sin(3x)$ satisfies $\frac{d^2y}{dx^2} + 9y = 0$

(c) $y = (A + Bx)e^{-x}$ satisfies $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

(d) $y = Ae^x + Be^{2x} + x$ satisfies $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = -2$

27. Reduce these second-order equations to first-order by substitution $v = \frac{dy}{dx}$:

(a) $\frac{d^2y}{dx^2} = \frac{dy}{dx}$

(b) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$

(c) $x\frac{d^2y}{dx^2} = \frac{dy}{dx}$

(d) $\frac{d^2y}{dx^2} = 1 + \left(\frac{dy}{dx}\right)^2$

28. Solve by direct integration (reducing order):

(a) $\frac{d^2y}{dx^2} = 6x$

(b) $\frac{d^2y}{dx^2} = e^{2x}$

(c) $\frac{d^2y}{dx^2} = \sin x$

(d) $\frac{d^2y}{dx^2} = x^2 + 2x - 1$

29. Solve with given initial conditions:

(a) $\frac{d^2y}{dx^2} = 12x^2$, $y(0) = 1$, $y'(0) = 2$

(b) $\frac{d^2y}{dx^2} = 2e^x$, $y(0) = 0$, $y'(0) = 1$

(c) $\frac{d^2y}{dx^2} = \cos x$, $y(0) = 1$, $y(\pi) = -1$

(d) $\frac{d^2y}{dx^2} = \frac{2}{x^3}$, $y(1) = 0$, $y'(1) = 1$ (for $x > 0$)

30. Physical interpretation:

(a) If $s(t)$ represents position and $\frac{d^2s}{dt^2} = -g$ (constant gravity), find $s(t)$ given initial position s_0 and velocity v_0.

(b) A particle moves so that its acceleration is $a = 6t$. Find position if $s(0) = 0$ and $v(0) = 2$.

(c) The acceleration of a falling object with air resistance is $a = g - kv^2$. Explain why this leads to a first-order equation for velocity.

Section G: Homogeneous Linear Second-Order Equations

31. Find the auxiliary equation and solve:

(a) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

(b) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 0$

(c) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = 0$

(d) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$

(e) $\frac{d^2y}{dx^2} - 4y = 0$

(f) $\frac{d^2y}{dx^2} + 16y = 0$

32. Classify the roots and write the general solution:

(a) $m^2 - 7m + 12 = 0$ (two distinct real roots)

(b) $m^2 + 6m + 9 = 0$ (repeated real root)

(c) $m^2 + 2m + 5 = 0$ (complex conjugate roots)

(d) $m^2 - 9 = 0$ (two distinct real roots)

(e) $m^2 + 1 = 0$ (pure imaginary roots)

(f) $m^2 - 4m + 4 = 0$ (repeated real root)

33. Solve these equations with initial conditions:

(a) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$, $y(0) = 1$, $y'(0) = 0$

(b) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$, $y(0) = 2$, $y'(0) = -1$

(c) $\frac{d^2y}{dx^2} + 4y = 0$, $y(0) = 1$, $y'(0) = 2$

(d) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 10y = 0$, $y(0) = 0$, $y'(0) = 1$

34. Analyze the behavior of solutions:

(a) For $\frac{d^2y}{dx^2} - k^2y = 0$ where $k > 0$, discuss the nature of solutions

(b) For $\frac{d^2y}{dx^2} + k^2y = 0$ where $k > 0$, describe the oscillatory behavior

(c) For $\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + a^2y = 0$, explain critical damping

(d) Sketch solution curves for different types of roots

35. Higher-order equations:

(a) Solve $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

(b) Find the general solution to $\frac{d^4y}{dx^4} - y = 0$

(c) Solve $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} - y = 0$

(d) Discuss the form of solutions for nth order linear homogeneous equations

Section H: Non-homogeneous Linear Second-Order Equations

36. Solve using the method of undetermined coefficients:

(a) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 6$

(b) $\frac{d^2y}{dx^2} + 4y = 8x$

(c) $\frac{d^2y}{dx^2} - \frac{dy}{dx} = e^x$

(d) $\frac{d^2y}{dx^2} + y = \sin x$

(e) $\frac{d^2y}{dx^2} - 4y = e^{2x}$

(f) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2$

37. Find particular integrals for these right-hand sides:

(a) Constant: $f(x) = k$

(b) Polynomial: $f(x) = ax^2 + bx + c$

(c) Exponential: $f(x) = Ae^{ax}$

(d) Trigonometric: $f(x) = A\cos(ax) + B\sin(ax)$

(e) Products: $f(x) = xe^x$, $f(x) = x\sin x$

38. Handle resonance cases:

(a) $\frac{d^2y}{dx^2} + 4y = \cos(2x)$ (resonance)

(b) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$ (resonance)

(c) $\frac{d^2y}{dx^2} + y = \sin x$ (resonance)

(d) Explain why we multiply by x in resonance cases

39. Complete solutions with initial conditions:

(a) $\frac{d^2y}{dx^2} + y = 2$, $y(0) = 1$, $y'(0) = 0$

(b) $\frac{d^2y}{dx^2} - y = x$, $y(0) = 0$, $y'(0) = 1$

(c) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x}$, $y(0) = 1$, $y'(0) = -1$

(d) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 10\cos x$, $y(0) = 0$, $y'(0) = 2$

40. Variation of parameters method:

(a) Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x}$ using variation of parameters
(b) Apply to $\frac{d^2y}{dx^2} + y = \sec x$
(c) Compare with undetermined coefficients where applicable
(d) When is variation of parameters necessary?

Section I: Applications of Second-Order Differential Equations

41. Simple harmonic motion:

(a) A mass on a spring satisfies $m\frac{d^2x}{dt^2} + kx = 0$. Solve for $x(t)$ given $x(0) = A$ and $\dot{x}(0) = 0$.
(b) Find the period and frequency of oscillation.
(c) If $m = 2$ kg and $k = 8$ N/m, and the mass is displaced 0.5 m from equilibrium and released, find $x(t)$.
(d) What is the maximum speed of the mass?

42. Damped harmonic motion:

(a) For $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$, classify motion based on discriminant.
(b) With $m = 1$, $c = 4$, $k = 3$, solve with $x(0) = 1$, $\dot{x}(0) = 0$ (overdamped).
(c) With $m = 1$, $c = 2$, $k = 1$, solve with $x(0) = 1$, $\dot{x}(0) = 0$ (critically damped).
(d) With $m = 1$, $c = 1$, $k = 2$, solve with $x(0) = 1$, $\dot{x}(0) = 0$ (underdamped).
(e) Sketch the three types of motion.

43. Forced oscillations:

(a) Solve $\frac{d^2x}{dt^2} + 4x = 8\cos(3t)$ with $x(0) = 0$, $\dot{x}(0) = 0$.
(b) Find the steady-state and transient solutions.
(c) Analyze resonance when forcing frequency equals natural frequency.
(d) With damping: $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 10\cos(2t)$, find the amplitude response.

44. Electrical circuits:

(a) An RLC circuit satisfies $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V(t)$. Explain each term.
(b) With $L = 1$ H, $R = 6$, $C = 0.1$ F, and $V = 10$ V (constant), solve for charge $q(t)$.
(c) Find the current $i(t) = \frac{dq}{dt}$.
(d) Analyze the circuit's natural frequency and damping ratio.

45. Beam deflection:

(a) A simply supported beam under uniform load satisfies $EI\frac{d^4y}{dx^4} = w$ where y is deflection.
(b) For a cantilever beam with point load at the end, $EI\frac{d^2y}{dx^2} = -M(x)$.
(c) Solve for the deflection curve with appropriate boundary conditions.
(d) Find the maximum deflection and its location.

46. Population dynamics:

(a) The predator-prey model gives coupled equations. For a simplified case where prey population oscillates: $\frac{d^2P}{dt^2} + k^2P = 0$, solve for $P(t)$.
(b) A population with inertia satisfies $\frac{d^2P}{dt^2} + a\frac{dP}{dt} + bP = cK$ where K is carrying capacity.
(c) Analyze stability of equilibrium solutions.
(d) Model boom-bust cycles in economics using similar equations.

Section J: Advanced Topics and Modeling

47. Systems of differential equations:
    (a) Solve the system: $\frac{dx}{dt} = 2x + y$, $\frac{dy}{dt} = x + 2y$
    (b) Convert $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ to a first-order system
    (c) Analyze equilibrium points and stability
    (d) Sketch phase portraits for different cases
48. Laplace transform method:
    (a) Use Laplace transforms to solve $\frac{d^2y}{dt^2} + 4y = 0$, $y(0) = 1$, $y'(0) = 2$
    (b) Solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^{-t}$ with zero initial conditions
    (c) Handle discontinuous forcing functions using unit step function
    (d) Apply to solve integro-differential equations
49. Series solutions:
    (a) Find power series solution to $\frac{d^2y}{dx^2} + xy = 0$ about $x = 0$
    (b) Solve $\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$ using series method
    (c) Identify convergence radius
    (d) Compare with known special functions
50. Boundary value problems:
    (a) Solve $\frac{d^2y}{dx^2} + \lambda y = 0$ with $y(0) = y(\pi) = 0$
    (b) Find eigenvalues and eigenfunctions
    (c) Apply to heat conduction: $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$
    (d) Use separation of variables for partial differential equations
51. Existence and uniqueness theorems:
    (a) State conditions for existence and uniqueness of solutions
    (b) Give examples where solutions don't exist or aren't unique
    (c) Discuss continuation of solutions
    (d) Analyze singular points and their classification
52. Stability analysis:
    (a) Define Lyapunov stability for differential equations
    (b) Analyze stability of linear systems using eigenvalues
    (c) Apply linearization to study nonlinear systems near equilibria
    (d) Use phase plane analysis for autonomous systems
53. Numerical methods for differential equations:
    (a) Apply Euler's method to $\frac{dy}{dx} = x + y$, $y(0) = 1$
    (b) Use improved Euler (Heun's) method for better accuracy
    (c) Implement Runge-Kutta methods
    (d) Discuss stability and convergence of numerical schemes
    (e) Apply to systems and higher-order equations

54. Green's functions:
    (a) Define Green's function for $\frac{d^2y}{dx^2} + y = f(x)$
    (b) Construct Green's function using boundary conditions
    (c) Apply to solve non-homogeneous boundary value problems
    (d) Relate to impulse response in physical systems
55. Advanced applications:
    (a) Model a double pendulum system with coupled equations
    (b) Analyze vibrations in mechanical systems with multiple degrees of freedom
    (c) Study wave propagation using $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$
    (d) Model chemical reaction kinetics with system of ODEs
    (e) Apply to epidemiological models (SIR model)
    (f) Study chaos in nonlinear differential equations
56. Integral equations and transforms:
    (a) Convert differential equations to integral equations
    (b) Use Fourier transforms for solving PDEs
    (c) Apply convolution theorem for Green's functions
    (d) Solve Volterra and Fredholm integral equations
    (e) Connect to probability and stochastic differential equations
57. Modeling project - Choose one real-world application:
    (a) Climate modeling: temperature variations over time
    (b) Epidemic spread: SIR or SEIR models with vaccination
    (c) Financial mathematics: Black-Scholes equation
    (d) Engineering: control systems and feedback
    (e) Biology: population genetics or enzyme kinetics
    (f) Physics: quantum harmonic oscillator or wave mechanics

    For your chosen application:
    (a) Derive the differential equation from first principles
    (b) Classify the equation and choose appropriate solution method
    (c) Solve analytically where possible, numerically otherwise
    (d) Interpret solutions in context of the physical problem
    (e) Validate model against real data or known results
    (f) Discuss limitations and possible extensions
    (g) Present findings with appropriate graphs and analysis

Section K: Problem-Solving Strategies and Review

58. Classification and strategy guide:
    (a) Create a flowchart for classifying differential equations
    (b) List solution methods for each type
    (c) Discuss when to use analytical vs. numerical methods
    (d) Identify common pitfalls and how to avoid them
59. Mixed practice problems:
    (a) $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$ (homogeneous)
    (b) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x} \sin x$
    (c) $(x + y)\frac{dy}{dx} = x - y$ (substitution method)
    (d) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ (Euler equation)
    (e) $\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}$ (polar substitution)
60. Verification and checking:
    (a) Methods for verifying solutions
    (b) Checking initial and boundary conditions
    (c) Dimensional analysis in physical problems
    (d) Limiting behavior and asymptotic analysis
61. Historical context and development:
    (a) Newton and Leibniz: origins of calculus and differential equations
    (b) Euler's contributions: linear equations and special functions
    (c) Lagrange and Hamilton: mechanics and variational principles
    (d) Modern developments: chaos theory and computer solutions
62. Connections to other areas:
    (a) Linear algebra: matrix methods for systems
    (b) Complex analysis: solutions in the complex plane
    (c) Fourier analysis: frequency domain methods
    (d) Probability: stochastic differential equations
    (e) Numerical analysis: computational solutions
63. Examination techniques:
    (a) Time management strategies for differential equation problems
    (b) Common examination question types and approaches
    (c) What to do when standard methods don't work
    (d) Partial credit strategies and clear presentation
    (e) Calculator and technology usage guidelines
64. Extended investigations:
    (a) Research a specific type of differential equation not covered in detail
    (b) Investigate historical development of a particular solution method

   (c) Study applications in your area of interest (physics, biology, economics)
   (d) Compare analytical and numerical solutions for a challenging problem
   (e) Explore modern research areas involving differential equations

65. Synthesis problems requiring multiple techniques:
   (a) A mass-spring system with time-varying parameters
   (b) Population model with seasonal variation and migration
   (c) RC circuit with switching and non-constant voltage
   (d) Heat conduction in a rod with temperature-dependent properties
   (e) Chemical reaction with multiple pathways and catalysis
   (f) Economic model with policy interventions and external shocks

66. Critical thinking questions:
   (a) When might a differential equation model be inappropriate?
   (b) How do you handle discontinuities in real-world applications?
   (c) What assumptions are inherent in linearization techniques?
   (d) How do numerical errors propagate in long-term simulations?
   (e) When is it better to use a simpler model vs. a more accurate one?

67. Future directions and advanced study:
   (a) Partial differential equations and field theory
   (b) Stochastic differential equations and noise
   (c) Delay differential equations and memory effects
   (d) Fractional differential equations and anomalous diffusion
   (e) Differential equations on manifolds and geometry
   (f) Computational fluid dynamics and numerical PDEs
   (g) Machine learning approaches to differential equations

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 250

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