A Level Pure Mathematics
Practice Test 4: Differential Equations

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Concepts and Classification

1. Explain with examples:
   (a) Order vs. degree of differential equations
   (b) Separable vs. non-separable equations
   (c) Transient vs. steady-state solutions
   (d) Bifurcation points in differential equations
   (e) Well-posed vs. ill-posed problems
   (f) Integral curves and solution families

2. Classify these equations:
   (a) $x\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = y$
   (b) $\frac{d^5y}{dx^5} - 3\frac{d^3y}{dx^3} + 2y = x^3$
   (c) $\arctan\left(\frac{dy}{dx}\right) + xy = 1$
   (d) $\left(\frac{d^2y}{dx^2}\right)^3 + x\frac{dy}{dx} - y^2 = 0$
   (e) $\frac{dy}{dx} + P(x)y = Q(x)y^{1/3}$ (generalized Bernoulli)
   (f) $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ (wave equation)

3. Verify these solutions:
   (a) $y = Ae^{3x} + Be^{-2x}$ satisfies $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$
   (b) $y = C_1x + C_2x\ln x$ satisfies $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$
   (c) $y = e^{-x}\sin x$ is a solution to $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$
   (d) $xy = C$ represents solutions to $x\frac{dy}{dx} + y = 0$

4. Find differential equations for:
   (a) $y = Ce^{-3x} + D$ (exponential with constant)
   (b) $y = C_1e^{2x}\cos x + C_2e^{2x}\sin x$ (complex exponentials)
   (c) $(x-h)^2 + (y-k)^2 = r^2$ (circles of radius r)

(d) $y = Cx^n$ where n is constant

5. Phase plane analysis:
   (a) For $\frac{dy}{dx} = (y-1)(y-3)$, find critical points
   (b) Classify stability of each equilibrium
   (c) Sketch phase portrait with flow directions
   (d) Describe basin of attraction for stable equilibria

Section B: Direct Integration and Elementary Methods

6. Integrate directly:
   (a) $\frac{dy}{dx} = 7x^3 - 4x + 3$
   (b) $\frac{dy}{dx} = 3e^{-x} - 2$
   (c) $\frac{dy}{dx} = \frac{6}{3x+2}$
   (d) $\frac{dy}{dx} = \csc^2(2x)$
   (e) $\frac{dy}{dx} = \frac{4x^3}{x^4+1}$
   (f) $\frac{dy}{dx} = (2x+1)e^{x^2+x}$

7. Solve with conditions:
   (a) $\frac{dy}{dx} = 15x^2 + 8x$, $y(1) = 8$
   (b) $\frac{dy}{dx} = 5e^{3x}$, $y(0) = 4$
   (c) $\frac{dy}{dx} = \cos(3x)$, $y(\pi/6) = 2$
   (d) $\frac{dy}{dx} = \frac{5}{x+4}$, $y(0) = \ln 4$ (for $x > -4$)
   (e) $\frac{dy}{dx} = 3x\sqrt{x^2+1}$, $y(0) = 7$

8. Higher derivatives:
   (a) $\frac{d^2y}{dx^2} = 12x^2 - 6$, $y(0) = 4$, $y'(0) = -3$
   (b) $\frac{d^2y}{dx^2} = 3e^x$, $y(0) = 2$, $y'(0) = 1$
   (c) $\frac{d^3y}{dx^3} = 18x$, $y(0) = 0$, $y'(0) = 3$, $y''(0) = -2$
   (d) $\frac{d^2y}{dx^2} = \sin(2x)$, $y(0) = 1$, $y(\pi/4) = 2$

9. Kinematics applications:
   (a) Acceleration $a = 10t - 8$. Find $v(t)$ and $s(t)$ if $v(0) = 3$, $s(0) = 1$.
   (b) Object dropped from 200m height. Find time to hit ground and impact velocity.
   (c) Spring equation $\frac{d^2x}{dt^2} = -16x$. Solve for $x(0) = 2$, $\dot{x}(0) = 4$.
   (d) Curve has second derivative $6x$. Find equation through $(1, 2)$ with slope 5.

10. Growth and decay:
   (a) Population grows at $\frac{dP}{dt} = 0.08P$. If $P(0) = 2000$, find $P(t)$ and doubling time.
   (b) Radioactive substance decays: $\frac{dN}{dt} = -0.03N$. Find half-life.
   (c) Investment: $\frac{dA}{dt} = 0.05A + 500$. Solve if $A(0) = 10000$.
   (d) Chemical concentration: $\frac{dc}{dt} = -kc^2$ where $k > 0$.

Section C: Separation of Variables

11. Solve by separation:
    (a) $\frac{dy}{dx} = 6xy^4$
    (b) $\frac{dy}{dx} = \frac{y^3}{x^2}$
    (c) $\frac{dy}{dx} = e^{3x-2y}$
    (d) $\frac{dy}{dx} = \frac{x^2 \cos x}{y^3}$
    (e) $\frac{dy}{dx} = \frac{\sin x}{\cos y}$
    (f) $\frac{dy}{dx} = \frac{x^2 y}{x^3+27}$
12. Initial value problems:
    (a) $\frac{dy}{dx} = 4xy$, $y(0) = 3$
    (b) $\frac{dy}{dx} = \frac{3y}{x}$, $y(1) = 8$ (for $x > 0$)
    (c) $\frac{dy}{dx} = \frac{x^4}{y^3}$, $y(0) = 2$
    (d) $\frac{dy}{dx} = y(5-y)$, $y(0) = 2$
    (e) $\frac{dy}{dx} = \frac{3x}{\sqrt{16-y^2}}$, $y(0) = 0$
13. Complex separable forms:
    (a) $(9+y^2)\frac{dy}{dx} = 4xy$
    (b) $\frac{dy}{dx} = \frac{ye^{4x}}{x^4+1}$
    (c) $\cos^2 y \frac{dy}{dx} = \sin(2x)$
    (d) $\frac{dy}{dx} = \frac{x^3(1+y^2)}{y(1+x^4)}$
    (e) $y^2 \ln y \frac{dy}{dx} = x^3$
14. Applications:
    (a) Malthusian growth: $\frac{dP}{dt} = 0.06P$, $P(0) = 1200$. When does population reach 5000?
    (b) Drug elimination: $\frac{dC}{dt} = -0.2C$. If $C(0) = 50$ mg/L, find time for concentration to drop to 10 mg/L.
    (c) Newton's law: $\frac{dT}{dt} = -k(T-22)$. Object cools from 95°C to 75°C in 5 minutes. Find k and time to reach 30°C.
    (d) Verhulst model: $\frac{dP}{dt} = rP(1-\frac{P}{K})$ with $r = 0.2$, $K = 2000$, $P(0) = 100$.
15. Separability tests:
    (a) $\frac{dy}{dx} = x^2y + xy^2$ (separable)
    (b) $\frac{dy}{dx} = x^2 + xy + y^2$ (not separable)
    (c) $\frac{dy}{dx} = \cos(x-y)$ (not separable)
    (d) $\frac{dy}{dx} = e^{2x-3y}$ (separable)
    (e) $\frac{dy}{dx} = \frac{xy}{x^2+1}$ (separable)

Section D: Linear First-Order Equations

16. Integrating factor method:
    (a) $\frac{dy}{dx} + 6y = e^{5x}$
    (b) $\frac{dy}{dx} - 4y = 3x^3$
    (c) $\frac{dy}{dx} + \frac{5y}{x} = x^4$ (for $x > 0$)
    (d) $\frac{dy}{dx} + y \tan x = \sec x \tan x$
    (e) $x\frac{dy}{dx} + 5y = x^3$
    (f) $\frac{dy}{dx} + 4xy = 2xe^{-2x^2}$
17. With initial conditions:
    (a) $\frac{dy}{dx} + 5y = 15e^{3x}$, $y(0) = 2$
    (b) $\frac{dy}{dx} - 3y = 9x$, $y(0) = 1$
    (c) $\frac{dy}{dx} + 4y = 12$, $y(0) = 0$
    (d) $\frac{dy}{dx} + \frac{3y}{x} = 6x$, $y(1) = 4$ (for $x > 0$)
18. Advanced forms:
    (a) $\frac{dy}{dx} + y \csc x = \cot x \csc x$
    (b) $(x^2 + 4)\frac{dy}{dx} + 2xy = x^2 + 4$
    (c) $\frac{dy}{dx} + \frac{4y}{x^2+1} = \frac{4x}{x^2+1}$
    (d) $x^4\frac{dy}{dx} + 3x^3y = x^6$ (for $x > 0$)
19. Engineering applications:
    (a) LC circuit: $L\frac{di}{dt} + \frac{q}{C} = V_0$ with $q = \int i\,dt$. Find $i(t)$.
    (b) Salt mixing: 300L tank, solution enters at 4 L/min (3 kg/L salt), exits at 4 L/min. Find salt concentration over time.
    (c) Compound interest: $\frac{dA}{dt} = 0.07A - 1500$ (7% rate, £1500 annual withdrawal).
    (d) Air resistance: $m\frac{dv}{dt} + cv = mg$ for $m = 2$ kg, $c = 0.1$, $g = 9.8$.
20. Method comparison:
    (a) Solve $\frac{dy}{dx} = 5xy + 5x$ by separation
    (b) Solve as linear: $\frac{dy}{dx} - 5xy = 5x$
    (c) Verify both give same result
    (d) Discuss advantages and computational complexity

Section E: Second-Order Homogeneous Equations

21. Characteristic equation method:
    (a) $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$
    (b) $\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 36y = 0$
    (c) $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 29y = 0$
    (d) $\frac{d^2y}{dx^2} + 64y = 0$
    (e) $\frac{d^2y}{dx^2} - 49y = 0$

(f) $\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 26y = 0$

22. Solution types:
    (a) $m^2 - 10m + 21 = 0$ (distinct real)
    (b) $m^2 + 14m + 49 = 0$ (repeated real)
    (c) $m^2 + 6m + 25 = 0$ (complex conjugate)
    (d) $m^2 - 100 = 0$ (distinct real)
    (e) $m^2 + 16 = 0$ (pure imaginary)
23. Boundary value problems:
    (a) $\frac{d^2y}{dx^2} - 11\frac{dy}{dx} + 28y = 0$, $y(0) = 3$, $y'(0) = 1$
    (b) $\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 25y = 0$, $y(0) = 1$, $y'(0) = -3$
    (c) $\frac{d^2y}{dx^2} + 36y = 0$, $y(0) = 0$, $y'(0) = 6$
    (d) $\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 37y = 0$, $y(0) = 2$, $y'(0) = 4$
24. Physical interpretation:
    (a) Free vibrations: mass-spring systems
    (b) Electrical oscillations: LC circuits
    (c) Damping effects: overdamped, critical, underdamped
    (d) Energy considerations and conservation
25. Higher-order:
    (a) $\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 8\frac{dy}{dx} - 4y = 0$
    (b) $\frac{d^4y}{dx^4} - 256y = 0$
    (c) Solution structure for general nth order

Section F: Non-homogeneous Second-Order Equations

26. Undetermined coefficients:
    (a) $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 24$
    (b) $\frac{d^2y}{dx^2} + 36y = 108x^2$
    (c) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 20y = e^{5x}$
    (d) $\frac{d^2y}{dx^2} + 25y = \cos(4x)$
    (e) $\frac{d^2y}{dx^2} - 25y = 5e^{-5x}$
    (f) $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = x^2 + 4$
27. Resonance cases:
    (a) $\frac{d^2y}{dx^2} + 36y = \sin(6x)$ (resonance)
    (b) $\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 20y = e^{4x}$ (resonance)
    (c) $\frac{d^2y}{dx^2} + 9y = \cos(3x)$ (resonance)
    (d) Why multiply particular solution by x?
28. Complete solutions:

(a) $\frac{d^2y}{dx^2} + 16y = 32$, $y(0) = 3$, $y'(0) = 0$
(b) $\frac{d^2y}{dx^2} - 16y = 32x$, $y(0) = 0$, $y'(0) = 4$
(c) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 9e^{-3x}$, $y(0) = 2$, $y'(0) = -3$

29. Trial function guide:
    (a) Polynomial forcing functions
    (b) Exponential forcing functions
    (c) Trigonometric forcing functions
    (d) Product and combination rules

30. Variation of parameters:
    (a) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = \frac{e^{2x}}{x}$
    (b) When standard methods fail
    (c) Wronskian determinant application

Section G: Physical Systems and Applications

31. Harmonic oscillators:
    (a) Mass-spring: $m\frac{d^2x}{dt^2} + kx = 0$ with $x(0) = 5$, $\dot{x}(0) = 0$, $m = 4$ kg, $k = 36$ N/m
    (b) Calculate period, frequency, and maximum velocity
    (c) Energy analysis: kinetic and potential
    (d) Simple pendulum approximation for small angles

32. Damped oscillations:
    (a) $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$ with $m = 1$, $c = 7$, $k = 10$ (overdamped)
    (b) Critical damping: $m = 1$, $c = 10$, $k = 25$ with $x(0) = 4$, $\dot{x}(0) = -5$
    (c) Underdamped: $m = 2$, $c = 6$, $k = 20$ with $x(0) = 3$, $\dot{x}(0) = 0$
    (d) Damping ratio and quality factor calculations

33. Forced systems:
    (a) $\frac{d^2x}{dt^2} + 49x = 98\cos(6t)$ with zero initial conditions
    (b) Steady-state amplitude and phase relationships
    (c) Resonance condition: $\frac{d^2x}{dt^2} + 36x = 72\cos(6t)$
    (d) Frequency response and bandwidth

34. Circuit analysis:
    (a) RLC series: $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V(t)$
    (b) Component values: $L = 0.8$ H, $R = 8$, $C = 0.125$ F, $V = 24$ V
    (c) Transient and steady-state behavior
    (d) Impedance and frequency response

35. Biological and economic models:
    (a) Population cycles: $\frac{d^2P}{dt^2} + a\frac{dP}{dt} + bP = cK$ where K is carrying capacity
    (b) Market dynamics: $\frac{d^2p}{dt^2} + \frac{dp}{dt} + p = D$ (price oscillations)
    (c) Stability conditions and equilibrium analysis
    (d) Parameter sensitivity and control

Section H: Advanced Techniques

36. Homogeneous first-order:
    (a) $\frac{dy}{dx} = \frac{4x+3y}{2x}$ (substitute $v = \frac{y}{x}$)
    (b) $\frac{dy}{dx} = \frac{x^2-2xy+y^2}{x^2}$
    (c) $(3x^2 + 2xy)dx + (x^2 + 3y^2)dy = 0$
    (d) Homogeneity test and transformation
37. Bernoulli equations:
    (a) $\frac{dy}{dx} + 5y = 3xy^4$ (substitute $v = y^{1-n}$)
    (b) $x\frac{dy}{dx} + 4y = 2y^3$
    (c) $\frac{dy}{dx} - \frac{4y}{x} = \frac{y^3}{x^4}$
38. Exact equations:
    (a) $(5x^4 + 4y)dx + (4x + 6y)dy = 0$
    (b) $(e^x \sin y + 3x^2)dx + (3y^2 + e^x \cos y)dy = 0$
    (c) Integrating factor determination
39. Special substitutions:
    (a) $\frac{d^2y}{dx^2} + \frac{4}{x}\frac{dy}{dx} = 0$ (let $v = \frac{dy}{dx}$)
    (b) $y\frac{d^2y}{dx^2} = 4\left(\frac{dy}{dx}\right)^2$
    (c) Euler-Cauchy: $x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 4y = 0$
40. Systems introduction:
    (a) $\frac{dx}{dt} = 5x + 3y$, $\frac{dy}{dt} = 3x + 5y$
    (b) Matrix formulation and eigenvalues
    (c) Phase plane sketching
    (d) Stability classification

Section I: Modeling and Synthesis

41. Major modeling project - select one:
    (a) Infectious disease spread with immunity loss
    (b) Ecosystem dynamics with competition
    (c) Chemical reaction networks with catalysis
    (d) Economic cycles with policy intervention
    (e) Structural engineering with multiple modes
    (f) Climate modeling with feedback mechanisms

    Complete analysis including:
    (a) Model derivation from physical laws
    (b) Mathematical classification and solution approach
    (c) Analytical solutions where feasible
    (d) Numerical methods for complex cases

(e) Parameter estimation and model validation
(f) Sensitivity analysis and uncertainty quantification
(g) Policy implications and predictions
(h) Model limitations and future improvements

42. Numerical methods:
    (a) Euler method for $\frac{dy}{dx} = y - x^2$, $y(0) = 2$
    (b) Modified Euler (midpoint method)
    (c) Runge-Kutta fourth-order method
    (d) Error estimation and convergence

43. Boundary problems:
    (a) $\frac{d^2y}{dx^2} + k^2y = 0$ with $y(0) = y(L) = 0$
    (b) Eigenfunction expansion
    (c) Heat equation separation of variables
    (d) Fourier series connections

44. Theory and existence:
    (a) Picard-Lindelöf theorem conditions
    (b) Uniqueness vs. non-uniqueness examples
    (c) Continuation of solutions
    (d) Blow-up phenomena

45. Integration and review:
    (a) Complete classification system
    (b) Solution strategy selection
    (c) Common mistakes and prevention
    (d) Historical perspective and modern developments
    (e) Interdisciplinary applications

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 250

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