A Level Pure Mathematics
Practice Test 2: Proof

Instructions:
Answer all questions. Show your working clearly.
Calculators may NOT be used in this test.
Time allowed: 2 hours

Section A: Direct Proof

1. Prove that the sum of two odd integers is always even.
2. Prove that if n is an even integer, then n^3 is even.
3. Prove that the product of two consecutive integers is always even.
4. Prove that for any integer n, the expression $3n^2 + 3n$ is always divisible by 3.
5. Given that p and q are rational numbers with $q \neq 0$, prove that $\frac{p}{q}$ is rational.
6. Prove that if $a \geq 0$ and $b \geq 0$, then $\sqrt{ab} \leq \frac{a+b}{2}$ (AM-GM inequality).
7. Prove that for any real numbers x and y, $(x-y)^2 \geq 0$ with equality if and only if $x = y$.
8. Prove that if a, b, and c are positive real numbers representing the sides of a triangle, then $a - b < c < a + b$.
9. Let $h(x) = -x^3 + 2x$. Prove that h is an odd function.
10. Prove that the function $k(x) = 5x - 2$ is strictly increasing on $\mathbb{R}$.

Section B: Proof by Contradiction

11. Prove that $\sqrt{7}$ is irrational.
12. Prove that there is no rational number whose square is 2.
13. Prove that $\sqrt{10}$ is irrational.
14. Prove that if n^3 is even, then n is even.
15. Prove that there is no greatest prime number.
16. Prove that if a and b are integers with $a^2 + b^2 = 7$, then both a and b cannot be even.
17. Prove that $\log_5 2$ is irrational.
18. Prove that there is no real number x such that $x^2 + 3x + 5 = 0$.
19. Prove that the equation $2x^2 - 3y^2 = 1$ has no integer solutions.
20. Prove that if p is an odd prime, then $p^2 + 2$ is not prime.

Section C: Mathematical Induction - Sequences and Series

21. Prove by induction that $3 + 6 + 9 + \ldots + 3n = \frac{3n(n+1)}{2}$ for all positive integers n.
22. Prove by induction that $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$ for all positive integers n.
23. Prove by induction that $1 + 4 + 7 + \ldots + (3n-2) = \frac{n(3n-1)}{2}$ for all positive integers n.
24. Prove by induction that $5 + 10 + 15 + \ldots + 5n = \frac{5n(n+1)}{2}$ for all positive integers n.
25. Prove by induction that $2 + 6 + 10 + \ldots + (4n-2) = 2n^2$ for all positive integers n.
26. Let $w_1 = 1$ and $w_{n+1} = 2w_n + 3$ for $n \geq 1$. Prove by induction that $w_n = 2^{n+1} - 3$ for all positive integers n.
27. Prove by induction that $\sum_{r=1}^{n} r \cdot 3^r = \frac{(2n-1)3^{n+1}+3}{4}$ for all positive integers n.
28. Prove by induction that $\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{3n+2}{4(n+1)}$ for all positive integers n.
29. The Lucas sequence is defined by $L_1 = 1$, $L_2 = 3$, and $L_{n+1} = L_n + L_{n-1}$ for $n \geq 2$. Prove by induction that $L_1 + L_2 + \ldots + L_n = L_{n+2} - 3$ for all $n \geq 1$.
30. Prove by induction that $\sum_{r=1}^{n} r^2(r+1) = \frac{n(n+1)(n+2)(3n+5)}{12}$ for all positive integers n.

Section D: Mathematical Induction - Inequalities

31. Prove by induction that $4^n \geq 3n + 1$ for all non-negative integers n.
32. Prove by induction that $2^n \geq n^2$ for all integers $n \geq 4$.
33. Prove by induction that $n! \geq 3^{n-2}$ for all integers $n \geq 3$.
34. Prove by induction that $(1-x)^n \geq 1 - nx$ for all real $0 \leq x \leq 1$ and all positive integers n.
35. Prove by induction that $\frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} < 1 - \frac{1}{n}$ for all integers $n \geq 2$.
36. Prove by induction that $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} \geq 2(\sqrt{n} - 1)$ for all integers $n \geq 2$.
37. Prove by induction that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots + \frac{1}{(2n-1)^2} < \frac{4}{3}$ for all positive integers n.
38. Prove by induction that $3^n \geq n^3$ for all integers $n \geq 4$.
39. Prove by induction that $\left(1 + \frac{1}{2n}\right)^n < 2$ for all positive integers n.
40. Prove by induction that for $n \geq 2$, $\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{3n} \geq 1$.

Section E: Mathematical Induction - Divisibility

41. Prove by induction that $n^3 + 5n$ is divisible by 6 for all positive integers n.
42. Prove by induction that $3^n - 1$ is divisible by 2 for all positive integers n.
43. Prove by induction that $7^n - 1$ is divisible by 6 for all positive integers n.
44. Prove by induction that $n^3 + 2n$ is divisible by 3 for all positive integers n.
45. Prove by induction that $9^n - 1$ is divisible by 8 for all positive integers n.
46. Prove by induction that $2^{3n} + 3^{2n+1}$ is divisible by 7 for all non-negative integers n.

47. Prove by induction that $10^n - 3^n$ is divisible by 7 for all positive integers n.
48. Prove by induction that $3^{2n} - 1$ is divisible by 8 for all positive integers n.
49. Prove by induction that $n^7 - n$ is divisible by 42 for all positive integers n.
50. Prove by induction that $11^n + 12^n$ is divisible by 23 for all odd positive integers n.

Section F: Deduction in Algebraic Manipulation

51. Given that $x + y = 3$ and $xy = -2$, find the value of $x^2 + y^2$.
52. If $a + b + c = 1$ and $a^2 + b^2 + c^2 = 5$, find the value of $ab + bc + ca$.
53. Given that α and β are roots of $x^2 - 3x + 1 = 0$, prove that:
    (a) $\alpha + \beta = 3$
    (b) $\alpha\beta = 1$
    (c) $\alpha^2 + \beta^2 = 7$
54. If $x + \frac{1}{x} = 4$, find expressions for:
    (a) $x^2 + \frac{1}{x^2}$
    (b) $x^3 + \frac{1}{x^3}$
    (c) $x^4 + \frac{1}{x^4}$
55. Prove that if $a + b + c = 0$, then $a^3 + b^3 + c^3 - 3abc = 0$.
56. Given that x, y, z are in arithmetic progression with common difference d, prove that $x+z = 2y$.
57. If $\cos A+\cos B+\cos C = 0$ and $\sin A+\sin B+\sin C = 0$, prove that $\cos 2A+\cos 2B+\cos 2C = 0$.
58. Prove that $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$.
59. Given that p, q, r are in geometric progression, prove that $\log p$, $\log q$, $\log r$ are in arithmetic progression.
60. If $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ are in arithmetic progression, prove that $\frac{b+c}{a}$, $\frac{c+a}{b}$, $\frac{a+b}{c}$ are in arithmetic progression.

Section G: Deduction in Geometric Reasoning

61. In triangle ABC, prove that the sum of the three interior angles is 180.
62. Prove that if a line is drawn parallel to one side of a triangle, it divides the other two sides proportionally.
63. Prove that the angle subtended by a diameter at any point on the circle is a right angle.
64. In triangle ABC, let G be the centroid. Prove that $AG : GM = 2 : 1$ where M is the midpoint of BC.
65. Prove that opposite angles of a cyclic quadrilateral are supplementary.
66. In a circle, prove that angles subtended by the same arc are equal.
67. Prove that the angle between a tangent and a chord equals the angle in the alternate segment.
68. In triangle ABC, prove that $a \sin B = b \sin A$ (part of the sine rule).
69. Prove that in any triangle, the three perpendicular bisectors of the sides meet at a single point.
70. Prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

Section H: Advanced Proof Techniques

71. Prove that the rational numbers are dense in the real numbers.

72. Prove that if $f(x) = mx + c$ where $m \neq 0$, then $f^{-1}(x) = \frac{x-c}{m}$.

73. Prove that the set of positive even integers has the same cardinality as the set of positive integers.

74. Use the pigeonhole principle to prove that among any 5 integers, at least two have the same remainder when divided by 4.

75. Prove that $\sqrt{3} + \sqrt{5}$ is irrational.

76. Prove that if p is prime and p divides a^2, then p divides a.

77. Prove that the quotient of two integers (with non-zero denominator) is rational.

78. Use strong induction to prove that every integer $n \geq 2$ has a prime divisor.

79. Prove that if $a_1, a_2, \ldots, a_n$ are positive real numbers, then:

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}}$$

(AM-HM inequality)

80. Prove or disprove: For all integers $n \geq 1$, $n^2 - n + 11$ is prime.

Section I: Proof Writing and Communication

81. Write a complete proof that for any triangle with sides a, b, c, the radius of the circumscribed circle is $R = \frac{abc}{4\Delta}$ where Δ is the area.

82. Prove that the equation $x^3 + y^3 = z^3$ has no positive integer solutions. (Provide a sketch of the proof strategy)

83. Let $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ (harmonic numbers). Prove that $H_{2^n} \geq 1 + \frac{n}{2}$ for all positive integers n.

84. Prove the identity: For real numbers a_1, a_2, b_1, b_2:

$$(a_1 + a_2)(b_1 + b_2) = a_1b_1 + a_1b_2 + a_2b_1 + a_2b_2$$

85. Consider the sequence defined by $d_1 = 3$, $d_2 = 5$, and $d_{n+2} = d_{n+1} + d_n$ for $n \geq 1$. Prove that $\gcd(d_n, d_{n+1}) = 1$ for all $n \geq 1$.

86. Prove that for any positive integer n, the number $5^{2n} - 3^n$ is divisible by 11.

87. Let $j : \mathbb{R} \setminus \{2\} \to \mathbb{R} \setminus \{1\}$ be defined by $j(x) = \frac{x+1}{x-2}$. Prove that j is bijective and find its inverse function.

88. Prove Euler's theorem: If $\gcd(a, n) = 1$, then $a^{\phi(n)} \equiv 1 \pmod{n}$ where $\phi(n)$ is Euler's totient function.

89. Prove that $\ln 2$ is irrational. (You may use properties of the natural logarithm and rational number arithmetic)

90. Write a proof by contradiction showing that there exist irrational numbers p and q such that $p + q$ is rational.

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

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