A Level Pure Mathematics
Practice Test 6: Differentiation

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Basic Differentiation - Polynomials

1. Differentiate these polynomial functions:
   (a) $f(x) = 6x^4 - 5x^3 + 7x^2 - 3x + 9$
   (b) $g(x) = 4x^6 + \frac{5}{6}x^4 - 7x + 15$
   (c) $h(x) = (x - 4)(x + 6)$
   (d) $k(x) = (5x + 2)^2$
   (e) $p(x) = x^5(x^3 - 1)$
   (f) $q(x) = \frac{x^7 - 3x^5 + 4x^3}{x^2}$
2. Find $\frac{dy}{dx}$ for:
   (a) $y = 7x^{-4} + 3x^{-1} - 6$
   (b) $y = \frac{4}{x^5} - \frac{7}{\sqrt{x}} + 5\sqrt{x}$
   (c) $y = 6\sqrt{x^9} + \frac{5}{x^4} - x^{-\frac{5}{6}}$
   (d) $y = (4x - \frac{5}{x})^2$
3. Find the gradient of these curves at the given points:
   (a) $y = 3x^4 - 7x^3 + 2x - 5$ at $x = 2$
   (b) $y = 5x^3 - 6x^2 + 8$ at $x = -2$
   (c) $y = \frac{4x^2 - 3}{x}$ at $x = 2$
   (d) $y = (x - 4)^3$ at $x = 5$
4. Find the equation of the tangent line to:
   (a) $y = 4x^3 - 3x^2 + 5x - 7$ at the point where $x = 1$
   (b) $y = 2x^2 - 8x + 6$ at the point $(3, -6)$
   (c) $y = x^3 - 12x$ at the point where the gradient is 15
   (d) $y = \frac{x^3}{6} - 4x + 2$ at the point where $x = 6$
5. Given that $f(x) = rx^3 + sx^2 + tx + u$ and $f'(x) = 15x^2 - 30x + 12$:
   (a) Find the values of r, s, and t
   (b) If $f(0) = 10$, find the value of u
   (c) Write the complete expression for $f(x)$
   (d) Find $f(2)$ and $f'(1)$

Section B: Differentiation of Special Functions

6. Differentiate these exponential and logarithmic functions:
   (a) $f(x) = 7e^x$
   (b) $g(x) = 6e^x + 5x^5$
   (c) $h(x) = x^4 e^x$
   (d) $k(x) = 4 \ln x$
   (e) $p(x) = x^4 \ln x$
   (f) $q(x) = \frac{\ln x}{x^3}$
7. Differentiate these trigonometric functions:
   (a) $f(x) = 6 \sin x - 4 \cos x$
   (b) $g(x) = 7 \sin x + 2 \cos x - x^5$
   (c) $h(x) = x^4 \sin x$
   (d) $k(x) = \frac{\cos x}{x^2}$
   (e) $p(x) = 4 \sec x$
   (f) $q(x) = \cot x$
8. Find $\frac{dy}{dx}$ for:
   (a) $y = e^{5x}$
   (b) $y = \ln(7x)$
   (c) $y = \sin(6x)$
   (d) $y = \cos(5x + 4)$
   (e) $y = e^{4x^2}$
   (f) $y = \ln(x^5 - 4)$
9. Differentiate using appropriate rules:
   (a) $f(x) = e^x \cot x$
   (b) $g(x) = x^5 \cos x$
   (c) $h(x) = \frac{e^x}{x^3}$
   (d) $k(x) = \frac{\sec x}{\tan x}$
   (e) $p(x) = (\ln x)^5$
   (f) $q(x) = \sqrt{\sec x}$
10. Find the derivatives of:
   (a) $f(x) = \sec^2 x$
   (b) $g(x) = \sin^6 x$
   (c) $h(x) = e^{\sec x}$
   (d) $k(x) = \ln(\sec x)$
   (e) $p(x) = (\sin x + \cos x)^5$
   (f) $q(x) = \tan^{-1} x$ (inverse tan)

Section C: Product Rule and Quotient Rule

11. Use the product rule to differentiate:
    (a) $f(x) = (x^5 + 3)(x^3 - 2)$
    (b) $g(x) = (5x - 2)(x^3 + 4x - 1)$
    (c) $h(x) = x^5 e^x$
    (d) $k(x) = (x - 4)\ln x$
    (e) $p(x) = \sec x \cot x$
    (f) $q(x) = x^5 \cos x$

12. Use the quotient rule to differentiate:
    (a) $f(x) = \frac{x^5-3}{x+4}$
    (b) $g(x) = \frac{5x-2}{x^2+3}$
    (c) $h(x) = \frac{e^x}{x^5}$
    (d) $k(x) = \frac{\ln x}{x-4}$
    (e) $p(x) = \frac{\sec x}{1+\tan x}$
    (f) $q(x) = \frac{x^5}{\sec x}$

13. Choose the most appropriate method to differentiate:
    (a) $f(x) = \frac{x^6+5x^4}{x^4}$
    (b) $g(x) = (x^4 - 3)(x + 5)$
    (c) $h(x) = \frac{x^5-3x^3+2}{x^5}$
    (d) $k(x) = x^4(x^2 + 3)^4$
    (e) $p(x) = \frac{(x+3)^5}{x^4}$
    (f) $q(x) = x^5\sqrt{x - 4}$

14. Given $f(x) = x^5$ and $g(x) = \tan x$:
    (a) Find $(fg)'(x)$ using the product rule
    (b) Find $(\frac{f}{g})'(x)$ using the quotient rule
    (c) Evaluate $(fg)'(\frac{\pi}{4})$
    (d) Evaluate $(\frac{f}{g})'(\frac{\pi}{3})$

15. Prove these differentiation rules:
    (a) Product rule: $(uv)' = u'v + uv'$
    (b) Quotient rule: $(\frac{u}{v})' = \frac{u'v-uv'}{v^2}$
    (c) Show that $(\frac{1}{v})' = -\frac{v'}{v^2}$
    (d) Verify that $(uvw)' = u'vw + uv'w + uvw'$

Section D: Chain Rule

16. Use the chain rule to differentiate:
    (a) $f(x) = (5x - 4)^6$
    (b) $g(x) = (x^4 - 3x + 2)^7$
    (c) $h(x) = \sqrt{4x^2 + 5}$
    (d) $k(x) = (6x + 5)^{-5}$
    (e) $p(x) = \sin(5x - 3)$
    (f) $q(x) = \cos(x^5)$

17. Find $\frac{dy}{dx}$ for:
    (a) $y = e^{5x+4}$
    (b) $y = \ln(6x - 2)$
    (c) $y = (x^4 - 3x)^8$
    (d) $y = \sec^2 x$
    (e) $y = \tan(e^x)$
    (f) $y = e^{\sec x}$

18. Differentiate these composite functions:
    (a) $f(x) = (e^x - 3)^6$
    (b) $g(x) = \ln(x^4 + 5x - 2)$
    (c) $h(x) = \tan(\ln x)$
    (d) $k(x) = e^{x \sec x}$
    (e) $p(x) = (\sin x - \cos x)^5$
    (f) $q(x) = \ln(\sec x)$

19. Use multiple rules to differentiate:
    (a) $f(x) = x^4(5x - 2)^6$
    (b) $g(x) = \frac{x^5}{(x-3)^5}$
    (c) $h(x) = x^5 \sin(5x)$
    (d) $k(x) = e^x \cos(5x)$
    (e) $p(x) = \frac{\ln x}{\sqrt{x^4+3}}$
    (f) $q(x) = \frac{(x^4+2)^5}{x^4}$

20. Find the second derivatives:
    (a) $f(x) = (x - 4)^7$
    (b) $g(x) = \sin(5x)$
    (c) $h(x) = e^{4x}$
    (d) $k(x) = \ln(x^5)$
    (e) $p(x) = x^5 e^x$
    (f) $q(x) = \tan x \tan x$

Section E: Stationary Points

21. Find the coordinates of stationary points for:
    (a) $f(x) = x^3 - 12x^2 + 36x + 7$
    (b) $g(x) = 5x^3 - 15x^2 + 10x - 3$
    (c) $h(x) = x^4 - 16x^2 + 32$
    (d) $k(x) = \frac{x^3}{x+2}$ for $x \neq -2$

22. Determine the nature of each stationary point using the second derivative test:
    (a) $f(x) = x^3 - 15x^2 + 63x - 5$
    (b) $g(x) = 5x^3 - 12x^2 - 45x + 2$
    (c) $h(x) = x^4 - 8x^2 + 16$
    (d) $k(x) = x^4 e^{-x}$

23. Find and classify all stationary points:
    (a) $f(x) = x^3 + 6x^2 - 15x + 8$
    (b) $g(x) = 5x^3 - 6x^2 - 45x + 4$
    (c) $h(x) = x^4 - 20x^2 + 100$
    (d) $k(x) = 3x + \frac{12}{x}$ for $x > 0$

24. For the function $f(x) = dx^3 + ex^2 + fx + g$:
    (a) Find the conditions on d, e, and f for the function to have two stationary points
    (b) If $f(x) = 4x^3 - 12x^2 + 12x + 3$, show it has no stationary points
    (c) Find the values of j for which $f(x) = x^3 - 12jx + 5$ has exactly one stationary point

25. Analyze the function $f(x) = \frac{x^3-27}{x}$:
    (a) Find the domain of $f(x)$
    (b) Find $f'(x)$ and locate stationary points
    (c) Determine the nature of stationary points
    (d) Find any asymptotes
    (e) Sketch the graph of $y = f(x)$

Section F: Rates of Change

26. A particle moves along a line with position $s(t) = 4t^3 - 15t^2 + 18t + 12$ meters at time t seconds.
    (a) Find the velocity $v(t)$ and acceleration $a(t)$
    (b) Find when the particle is at rest
    (c) Calculate the velocity and acceleration at $t = 2$
    (d) Determine when the acceleration is zero
    (e) Find the displacement between $t = 0$ and $t = 4$

27. The area of a circle is $A = \pi r^2$. If the radius increases at a rate of 5 cm/s:
    (a) Find the rate of change of area when $r = 8$ cm
    (b) Express $\frac{dA}{dt}$ in terms of r and $\frac{dr}{dt}$
    (c) When is the area increasing at 400π cm²/s?

(d) Find the rate of change of circumference when $r = 12$ cm

28. A ladder 12 meters long leans against a vertical wall. The bottom slides away at 3 m/s.
    (a) Set up the relationship between distances
    (b) Find how fast the top slides down when the bottom is 9m from the wall
    (c) Find the rate of change of the angle with the ground
    (d) When is the top sliding down fastest?

29. Water flows out of a spherical tank at 5 m^3/min. The tank has radius 6m.
    (a) Express the volume in terms of height h of water
    (b) Find how fast the water level drops when $h = 4$m
    (c) Find the rate of change of surface area when $h = 8$m
    (d) When is the water level dropping fastest?

30. The temperature of a cooling object follows $T(t) = 20 + 60e^{-0.05t}$ degrees Celsius at time t minutes.
    (a) Find the cooling rate $\frac{dT}{dt}$
    (b) Calculate the temperature and cooling rate after 10 minutes
    (c) When is the object cooling at 2°C per minute?
    (d) Express the cooling rate as a percentage of excess temperature

Section G: Optimization Problems

31. A farmer has 500m of fencing to enclose a rectangular field with three dividers parallel to one side.
    (a) Express the area in terms of one variable
    (b) Find the dimensions for maximum area
    (c) Calculate the maximum area
    (d) Verify this is a maximum using the second derivative

32. An open-top cylindrical container has volume 100 m^3. The material for the base costs £10/m^2 and sides cost £6/m^2.
    (a) Express the cost in terms of the radius
    (b) Find dimensions for minimum cost
    (c) Calculate the minimum cost
    (d) Find the ratio of height to radius

33. A factory's efficiency function is $E(x) = -x^3 + 21x^2 - 120x + 400$ percent, where x is hours worked per day.
    (a) Find the working hours for maximum and minimum efficiency
    (b) Calculate the maximum efficiency
    (c) Find the marginal efficiency function
    (d) Determine the optimal working hours

34. A rectangular frame surrounds a picture. The picture area is 200 cm^2 and the frame is 3cm wide on all sides.

(a) Express the frame area in terms of picture width
(b) Find picture dimensions for minimum frame area
(c) Calculate the minimum frame area
(d) Find the ratio of picture length to width

35. A right circular cylinder is inscribed in a sphere of radius 10 cm. Find dimensions to maximize volume.

(a) Express volume in terms of cylinder radius
(b) Find the critical points
(c) Determine optimal cylinder radius and height
(d) Calculate maximum volume
(e) Verify this gives a maximum

Section H: Implicit Differentiation and Related Rates

36. Find $\frac{dy}{dx}$ using implicit differentiation:

(a) $x^2 + y^2 = 64$
(b) $x^2 + 5xy + y^2 = 25$
(c) $x^3 + y^3 = 18xy$
(d) $\sec(xy) = 2x + y$
(e) $e^{x-y} = x + 2y$
(f) $\ln(x + 2y) = xy$

37. Find the equation of the tangent to these curves at the given points:

(a) $x^2 + y^2 = 45$ at $(3, 6)$
(b) $x^2 + 2xy + y^2 = 32$ at $(4, 2)$
(c) $x^3 + y^3 = 28$ at $(3, 1)$
(d) $xe^y = 8$ at $(4, \ln 2)$

38. Use implicit differentiation to find $\frac{d^2y}{dx^2}$:

(a) $x^2 + y^2 = 36$
(b) $xy = 16$
(c) $x^2 - y^2 = 25$

39. Two trains start from stations 100 km apart. Train A travels north at 120 km/h, Train B travels east at 90 km/h.

(a) Express the distance between trains as a function of time
(b) Find how fast they're separating after 0.75 hours
(c) When are they separating at 180 km/h?
(d) Find the minimum distance between them

40. A helium balloon is inflated so its volume increases at 150 cm³/s. Find the rate of increase of:

(a) Radius when $r = 6$ cm
(b) Surface area when $r = 10$ cm
(c) Diameter when volume is 8000 cm³
(d) The rate when surface area is 600π cm²

Section I: Advanced Applications

41. A cathedral window has the shape of a rectangle topped by a regular hexagon, with total perimeter 30m.
    (a) Find dimensions to maximize the area
    (b) Calculate the maximum area
    (c) Find the optimal ratio of rectangle height to width
    (d) Determine what fraction of area is rectangular

42. The moment of inertia of a rectangular beam is proportional to w^3d where w is width and d is depth. A beam is cut from a circular log of radius 20 cm.
    (a) Express moment of inertia in terms of width w
    (b) Find dimensions for maximum moment of inertia
    (c) Calculate the ratio $\frac{w}{d}$ for optimal beam
    (d) Compare with beam of square cross-section

43. A chemical reaction rate follows $R(t) = \frac{Dt^4}{(t+4)^5}$ mol/s where t is time in seconds.
    (a) Find when reaction rate is maximum
    (b) If peak rate is 8 mol/s, find D
    (c) Calculate the rate of change at $t = 4$
    (d) Find when reaction rate is decreasing fastest
    (e) Determine the time for quarter-peak reaction rate

44. A right triangle is inscribed in a parabola $y = 16 - x^2$ with vertices at $(a, 16 - a^2)$, $(-a, 16 - a^2)$, and $(0, 0)$.
    (a) Express the triangle area in terms of a
    (b) Find a for maximum area
    (c) Calculate the maximum area
    (d) Show the optimal triangle has specific angles

45. A courier company's hourly revenue is $R(v) = 0.4v^3 - 4.8v^2 + 16v + 120$ for delivering at v packages per hour.
    (a) Find the marginal revenue function
    (b) Determine the delivery rate for maximum revenue
    (c) Calculate the maximum hourly revenue
    (d) Find when marginal revenue equals zero
    (e) Graph the revenue function and interpret economically

46. Two positive numbers have difference 24. Find the numbers that:
    (a) Minimize their product
    (b) Maximize the sum of their squares
    (c) Minimize the product of their cube roots
    (d) Maximize $x^2 + y^4$ where $x - y = 24$
    (e) Explain why the answers differ

47. A communications satellite follows a path described by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (hyperbola).

(a) Find the slope at any point (x, y) on the path
(b) Find points where the tangent has slope m
(c) Calculate the slope at the point $(a \sec\theta, b \tan\theta)$
(d) Find the angle between asymptotes and tangent
(e) Derive conditions for specific orbital characteristics

48. Design a calculus optimization problem from environmental science:
    (a) Define your environmental scenario and variables clearly
    (b) Set up the objective function and constraints
    (c) Use differentiation to find optimal solutions
    (d) Verify your solution makes environmental sense
    (e) Discuss ecological factors not captured in your model

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

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