A Level Pure Mathematics
Practice Test 5: Differential Equations

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Theory and Classification

1. Define and illustrate:
   (a) Slope fields and isoclines
   (b) Continuous dependence on initial conditions
   (c) Local vs. global solutions
   (d) Conservative vs. dissipative systems
   (e) Limit cycles and periodic solutions
   (f) Separatrices in phase space

2. Classify by type and properties:
   (a) $\frac{d^2y}{dx^2} + y\left(\frac{dy}{dx}\right)^2 = x^2$
   (b) $\frac{d^6y}{dx^6} - 4\frac{d^4y}{dx^4} + 3y = \cos x$
   (c) $e^{\frac{dy}{dx}} + xy^2 = 1$
   (d) $\left(\frac{d^3y}{dx^3}\right)^2 + 2\frac{dy}{dx} - y = 0$
   (e) $\frac{dy}{dx} + Q(x)y = R(x)y^{2/3}$ (fractional Bernoulli)
   (f) $\frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ (damped wave)

3. Confirm these solution relationships:
   (a) $y = C_1e^{4x} + C_2e^{-x}$ satisfies $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 0$
   (b) $y = x^2(C_1 + C_2 \ln x)$ satisfies $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 0$
   (c) $y = e^{3x}\cos(2x)$ is a particular solution to $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0$
   (d) $x^2 + y^2 = C$ gives solutions to $x + y\frac{dy}{dx} = 0$

4. Generate differential equations from:
   (a) $y = Ce^{4x} + De^{-x} + 2$ (mixed exponential with constant)
   (b) $y = C_1e^{-x}\cos(3x) + C_2e^{-x}\sin(3x)$ (damped oscillation)
   (c) $y^2 = 4p(x - h)$ (parabolas with vertex $(h, 0)$)

   (d) $y = \frac{C}{x^2}$ (inverse square law)

5. Dynamical systems analysis:
   (a) For $\frac{dy}{dx} = y^2 - 2y - 3$, locate equilibrium points
   (b) Determine linear stability of each equilibrium
   (c) Construct phase portrait with flow arrows
   (d) Identify separatrices and domains of attraction

Section B: Integration Methods

6. Solve by integration:
   (a) $\frac{dy}{dx} = 8x^3 - 6x^2 + 1$
   (b) $\frac{dy}{dx} = 4e^{-2x} + 3$
   (c) $\frac{dy}{dx} = \frac{8}{4x-1}$
   (d) $\frac{dy}{dx} = \sec^2(4x) + 1$
   (e) $\frac{dy}{dx} = \frac{5x^4}{x^5+32}$
   (f) $\frac{dy}{dx} = (3x^2 - 1)e^{x^3-x}$

7. Apply initial conditions:
   (a) $\frac{dy}{dx} = 18x^2 - 12x$, $y(1) = 6$
   (b) $\frac{dy}{dx} = 6e^{4x}$, $y(0) = 5$
   (c) $\frac{dy}{dx} = \sin(4x)$, $y(\pi/8) = 3$
   (d) $\frac{dy}{dx} = \frac{6}{x+5}$, $y(0) = \ln 5$ (for $x > -5$)
   (e) $\frac{dy}{dx} = 4x\sqrt{x^2+9}$, $y(0) = 10$

8. Multiple derivatives:
   (a) $\frac{d^2y}{dx^2} = 20x^3 - 12$, $y(0) = 5$, $y'(0) = -4$
   (b) $\frac{d^2y}{dx^2} = 4e^{2x}$, $y(0) = 3$, $y'(0) = 0$
   (c) $\frac{d^3y}{dx^3} = 30x^2$, $y(0) = 1$, $y'(0) = 4$, $y''(0) = -3$
   (d) $\frac{d^2y}{dx^2} = \cos(3x)$, $y(0) = 2$, $y(\pi/6) = 1$

9. Mechanics problems:
   (a) Acceleration $a = 12t - 10$. Find $v(t)$ and $s(t)$ if $v(0) = 4$, $s(0) = -1$.
   (b) Projectile launched upward at 40 m/s from 50m height. Find maximum height and time to ground.
   (c) Oscillator: $\frac{d^2x}{dt^2} = -36x$. Solve for $x(0) = 1$, $\dot{x}(0) = 6$.
   (d) Curve with curvature proportional to x^2. Find equation through $(0, 3)$ with slope -2.

10. Rate applications:
   (a) Cell division: $\frac{dN}{dt} = 0.12N$. If $N(0) = 500$, find time for $N = 2000$.
   (b) Isotope decay: $\frac{dM}{dt} = -0.04M$. Find percentage remaining after 25 years.
   (c) Savings: $\frac{dA}{dt} = 0.08A + 800$. Solve if $A(0) = 5000$.
   (d) Pollution removal: $\frac{dC}{dt} = -k\sqrt{C}$ where $k > 0$ and C is concentration.

Section C: Separable Equations

11. Separate and solve:
    (a) $\frac{dy}{dx} = 8xy^5$
    (b) $\frac{dy}{dx} = \frac{y^4}{x^3}$
    (c) $\frac{dy}{dx} = e^{4x-3y}$
    (d) $\frac{dy}{dx} = \frac{x^3 \sin x}{y^4}$
    (e) $\frac{dy}{dx} = \frac{\cos x}{\sin y}$
    (f) $\frac{dy}{dx} = \frac{x^3 y}{x^4+81}$
12. Particular solutions:
    (a) $\frac{dy}{dx} = 6xy$, $y(0) = 4$
    (b) $\frac{dy}{dx} = \frac{4y}{x}$, $y(1) = 12$ (for $x > 0$)
    (c) $\frac{dy}{dx} = \frac{x^5}{y^4}$, $y(0) = 3$
    (d) $\frac{dy}{dx} = y(6-y)$, $y(0) = 1$
    (e) $\frac{dy}{dx} = \frac{4x}{\sqrt{25-y^2}}$, $y(0) = 0$
13. Advanced separable:
    (a) $(16+y^2)\frac{dy}{dx} = 5xy$
    (b) $\frac{dy}{dx} = \frac{ye^{5x}}{x^5+1}$
    (c) $\sin^3 y \frac{dy}{dx} = \cos(3x)$
    (d) $\frac{dy}{dx} = \frac{x^4(1+y^2)}{y(1+x^5)}$
    (e) $y^3 \ln y \frac{dy}{dx} = x^4$
14. Real applications:
    (a) Bacteria growth: $\frac{dB}{dt} = 0.15B$, $B(0) = 2000$. When does $B = 10000$?
    (b) Medical tracer: $\frac{dT}{dt} = -0.25T$. If half-life is observed, verify decay constant.
    (c) Heat transfer: $\frac{dT}{dt} = -k(T-15)$. Object cools 100°C to 80°C in 3 minutes. Find time to 25°C.
    (d) Competition model: $\frac{dP}{dt} = rP(1 - \frac{P}{K} - \frac{Q}{L})$ where Q is competitor.
15. Classification check:
    (a) $\frac{dy}{dx} = xy + x^2y$ (separable)
    (b) $\frac{dy}{dx} = x + y + xy$ (not separable)
    (c) $\frac{dy}{dx} = \sin(x+2y)$ (not separable)
    (d) $\frac{dy}{dx} = e^{3x+2y}$ (separable)
    (e) $\frac{dy}{dx} = \frac{x^2y^2}{x+1}$ (separable)

Section D: First-Order Linear

16. Integrating factors:
    (a) $\frac{dy}{dx} + 7y = e^{6x}$
    (b) $\frac{dy}{dx} - 5y = 4x^4$
    (c) $\frac{dy}{dx} + \frac{6y}{x} = x^5$ (for $x > 0$)
    (d) $\frac{dy}{dx} + 2y\cos x = \sin x \cos x$
    (e) $x\frac{dy}{dx} + 6y = x^4$
    (f) $\frac{dy}{dx} + 5xy = 3xe^{-5x^2/2}$
17. Initial conditions:
    (a) $\frac{dy}{dx} + 6y = 18e^{4x}$, $y(0) = 3$
    (b) $\frac{dy}{dx} - 4y = 12x$, $y(0) = 2$
    (c) $\frac{dy}{dx} + 5y = 15$, $y(0) = 0$
    (d) $\frac{dy}{dx} + \frac{4y}{x} = 8x$, $y(1) = 6$ (for $x > 0$)
18. Complex forms:
    (a) $\frac{dy}{dx} + 2y\sec x = \tan x \sec x$
    (b) $(x^2+9)\frac{dy}{dx} + 2xy = x^2 + 9$
    (c) $\frac{dy}{dx} + \frac{5y}{x^2+1} = \frac{5x}{x^2+1}$
    (d) $x^5\frac{dy}{dx} + 4x^4y = x^7$ (for $x > 0$)
19. Applications:
    (a) RLC circuit: $R\frac{di}{dt} + \frac{i}{C} = \frac{dV}{dt}$ with $V = V_0\sin(t)$.
    (b) Dilution tank: 400L tank, pure water enters at 5 L/min, mixture exits at 5 L/min. Find salt concentration.
    (c) Retirement fund: $\frac{dF}{dt} = 0.09F - 2000$ (9
    (d) Drag force: $m\frac{dv}{dt} + kv^2 = mg$ linearized for small velocities.
20. Method validation:
    (a) Solve $\frac{dy}{dx} = 6xy + 6x$ by separation
    (b) Solve as linear: $\frac{dy}{dx} - 6xy = 6x$
    (c) Confirm equivalence of solutions
    (d) Analyze which method is more efficient

Section E: Homogeneous Second-Order

21. Characteristic roots:
    (a) $\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 18y = 0$
    (b) $\frac{d^2y}{dx^2} + 14\frac{dy}{dx} + 49y = 0$
    (c) $\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 40y = 0$
    (d) $\frac{d^2y}{dx^2} + 81y = 0$
    (e) $\frac{d^2y}{dx^2} - 64y = 0$

(f) $\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 37y = 0$

22. Root types and general solutions:

(a) $m^2 - 12m + 27 = 0$ (distinct real)
(b) $m^2 + 16m + 64 = 0$ (repeated real)
(c) $m^2 + 8m + 41 = 0$ (complex conjugate)
(d) $m^2 - 121 = 0$ (distinct real)
(e) $m^2 + 25 = 0$ (pure imaginary)

23. Initial value solutions:

(a) $\frac{d^2y}{dx^2} - 13\frac{dy}{dx} + 36y = 0$, $y(0) = 4$, $y'(0) = 2$
(b) $\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 36y = 0$, $y(0) = 2$, $y'(0) = -4$
(c) $\frac{d^2y}{dx^2} + 49y = 0$, $y(0) = 0$, $y'(0) = 7$
(d) $\frac{d^2y}{dx^2} - 14\frac{dy}{dx} + 50y = 0$, $y(0) = 3$, $y'(0) = 5$

24. Behavior analysis:

(a) Exponential growth and decay patterns
(b) Oscillatory solutions with amplitude and phase
(c) Critical damping and overdamping
(d) Stability and long-term behavior

25. Higher-order extensions:

(a) $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$
(b) $\frac{d^4y}{dx^4} - 1296y = 0$
(c) General solution patterns for order n

Section F: Non-homogeneous Second-Order

26. Particular integrals:

(a) $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 30$
(b) $\frac{d^2y}{dx^2} + 49y = 147x^2$
(c) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 30y = e^{6x}$
(d) $\frac{d^2y}{dx^2} + 36y = \sin(5x)$
(e) $\frac{d^2y}{dx^2} - 36y = 6e^{6x}$
(f) $\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 25y = x^2 + 5$

27. Resonance handling:

(a) $\frac{d^2y}{dx^2} + 49y = \cos(7x)$ (resonance)
(b) $\frac{d^2y}{dx^2} - 11\frac{dy}{dx} + 30y = e^{5x}$ (resonance)
(c) $\frac{d^2y}{dx^2} + 16y = \sin(4x)$ (resonance)
(d) Modification principle for resonant forcing

28. Complete IVP:

(a) $\frac{d^2y}{dx^2} + 25y = 50$, $y(0) = 4$, $y'(0) = 0$
(b) $\frac{d^2y}{dx^2} - 25y = 50x$, $y(0) = 0$, $y'(0) = 5$
(c) $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 16e^{-4x}$, $y(0) = 3$, $y'(0) = -6$

29. Solution strategies:

    (a) Constant forcing: constant particular solution
    (b) Polynomial forcing: polynomial trial solutions
    (c) Exponential forcing: exponential trial solutions
    (d) Trigonometric forcing: sine-cosine combinations

30. Advanced methods:

    (a) Variation of parameters: $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x}$
    (b) Wronskian determinant calculation
    (c) Green's function concepts

Section G: Systems and Applications

31. Mechanical systems:

    (a) Spring-mass: $m\frac{d^2x}{dt^2} + kx = 0$ with $x(0) = 6$, $\dot{x}(0) = 0$, $m = 5$ kg, $k = 45$ N/m
    (b) Calculate natural frequency and total energy
    (c) Compound pendulum with moment of inertia
    (d) Coupled oscillators with two masses and three springs

32. Damped vibrations:

    (a) $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$ with $m = 1$, $c = 8$, $k = 12$ (overdamped)
    (b) Critical damping: $m = 1$, $c = 12$, $k = 36$ with $x(0) = 5$, $\dot{x}(0) = -6$
    (c) Underdamped: $m = 3$, $c = 9$, $k = 30$ with $x(0) = 4$, $\dot{x}(0) = 0$
    (d) Damping coefficient and Q-factor determination

33. External forcing:

    (a) $\frac{d^2x}{dt^2} + 64x = 128\cos(7t)$ with zero initial conditions
    (b) Amplitude and phase of steady-state response
    (c) Resonance: $\frac{d^2x}{dt^2} + 49x = 98\cos(7t)$
    (d) Beat phenomena and envelope modulation

34. Electrical networks:

    (a) RLC circuit: $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V(t)$
    (b) Values: $L = 1.2$ H, $R = 10$, $C = 0.08$ F, $V = 30$ V
    (c) Impedance matching and power transfer
    (d) Filter characteristics and cutoff frequencies

35. Biological models:

    (a) Predator-prey: $\frac{dx}{dt} = ax - bxy$, $\frac{dy}{dt} = -cy + dxy$
    (b) Linearization around equilibrium points
    (c) Phase plane trajectories and stability
    (d) Harvesting effects and sustainable yields

Section H: Advanced Topics

36. Special first-order types:
    (a) Homogeneous: $\frac{dy}{dx} = \frac{5x+4y}{3x}$ (substitute $v = \frac{y}{x}$)
    (b) Bernoulli: $\frac{dy}{dx} + 6y = 4xy^5$ (substitute $v = y^{1-n}$)
    (c) Exact: $(6x^5 + 5y)dx + (5x + 8y)dy = 0$
    (d) Riccati: $\frac{dy}{dx} = y^2 + \frac{1}{x^2}$ (special cases)
37. Reduction methods:
    (a) $\frac{d^2y}{dx^2} + \frac{5}{x}\frac{dy}{dx} = 0$ (substitute $v = \frac{dy}{dx}$)
    (b) $y\frac{d^2y}{dx^2} = 5\left(\frac{dy}{dx}\right)^2$
    (c) Euler: $x^2\frac{d^2y}{dx^2} + 7x\frac{dy}{dx} + 9y = 0$
    (d) Missing variable techniques
38. Systems of equations:
    (a) $\frac{dx}{dt} = 6x + 4y$, $\frac{dy}{dt} = 4x + 6y$
    (b) Eigenvalue analysis and diagonalization
    (c) Node, saddle, and spiral classifications
    (d) Stability criteria and Lyapunov functions
39. Transform methods:
    (a) Laplace transforms for $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = \delta(t)$
    (b) Impulse response and transfer functions
    (c) Convolution and Green's functions
    (d) Fourier methods for periodic forcing
40. Boundary value problems:
    (a) $\frac{d^2y}{dx^2} + \mu^2 y = 0$ with $y(0) = y(\pi) = 0$
    (b) Eigenvalue spectrum and orthogonal eigenfunctions
    (c) Heat equation and separation of variables
    (d) Sturm-Liouville operators

Section I: Comprehensive Project

41. Major modeling study - choose one:
    (a) Epidemic dynamics with vaccination and mutation
    (b) Ecosystem modeling with multiple species interactions
    (c) Climate feedback systems and tipping points
    (d) Financial markets with stochastic volatility
    (e) Neural network dynamics and learning
    (f) Structural dynamics under seismic loading

    Complete analysis:

    (a) Physical/biological principle formulation

(b) Mathematical model derivation and assumptions
(c) Dimensional analysis and nondimensionalization
(d) Analytical solutions where possible
(e) Numerical simulation for nonlinear cases
(f) Parameter estimation from data
(g) Sensitivity analysis and robustness
(h) Model validation and predictive capability
(i) Policy recommendations and future research

42. Computational methods:

(a) Adaptive Runge-Kutta for $\frac{dy}{dx} = \sin(xy)$, $y(0) = 1$
(b) Stiff equation solvers
(c) Boundary value problem shooting methods
(d) Finite difference and finite element approaches

43. Advanced theory:

(a) Existence theorems and uniqueness conditions
(b) Continuation and bifurcation theory
(c) Chaos and strange attractors
(d) Perturbation methods and multiple scales

44. Synthesis:

(a) Complete taxonomy of differential equations
(b) Solution methodology decision tree
(c) Historical development and modern applications
(d) Connections to other mathematical disciplines
(e) Future directions in differential equations research

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 250

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