A Level Pure Mathematics
Practice Test 6: Algebra and Functions

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Algebraic Manipulation

1. Simplify these expressions:
   (a) $\frac{x^2-64}{x^2+16x+64}$
   (b) $\frac{7x^2-28}{x^2+3x-18}$
   (c) $\frac{x^3-343}{x^2+7x+49}$
   (d) $\frac{x^4-2401}{x^2-7x-98}$

2. Factorize completely:
   (a) $x^3 + 18x^2 + 108x + 216$
   (b) $512x^3 - 729$
   (c) $x^{10} - 1024$
   (d) $x^{20} - 1$
   (e) $x^4 + 20x^2 + 100$
   (f) $x^3 - 7x^2 + 14x - 56$

3. Express as single fractions in simplest form:
   (a) $\frac{7}{x-4} - \frac{5}{x+6}$
   (b) $\frac{6x}{x^2-36} + \frac{4}{x-6}$
   (c) $\frac{4x-1}{x^2-5x+6} - \frac{2x+3}{x^2+2x-15}$
   (d) $\frac{6}{x-2} + \frac{5}{x+4} - \frac{7}{x^2+2x-8}$

4. Use the binomial theorem to expand:
   (a) $(7x + 3)^4$
   (b) $(4x - \frac{1}{3x})^5$
   (c) $(5 + 2x)^9$, and find the coefficient of x^7
   (d) Find the coefficient of x^0 in the expansion of $(x^6 + \frac{1}{x^4})^{15}$

5. Simplify using laws of indices:
   (a) $\frac{13^{4x-2} \cdot 169^{2x+1}}{2197^{x+3}}$

(b) $\frac{1024^{x-1} \cdot 512^{3x}}{256^{2x+2}}$

(c) $(x^{\frac{7}{8}})^{\frac{8}{9}} \cdot x^{-\frac{5}{6}}$

(d) $\frac{(10x)^3 \cdot (4x^5)^2}{40x^{12}}$

Section B: Linear and Quadratic Equations

6. Solve these equations:

   (a) $\frac{7x+3}{8} - \frac{5x-2}{6} = \frac{1}{4}$

   (b) $\frac{6x}{x-3} = \frac{8}{x+5}$

   (c) $\sqrt{7x+2} = 5x - 8$

   (d) $\frac{6}{x+5} - \frac{4}{x-3} = \frac{1}{6}$

7. Solve these quadratic equations, leaving answers in exact form where appropriate:

   (a) $8x^2 - 15x + 6 = 0$

   (b) $x^2 - 16x + 19 = 0$

   (c) $7x^2 = 9x + 4$

   (d) $(7x - 4)^2 = 5(4x - 1)$

8. For the quadratic equation $6x^2 - (5k + 2)x + 4k - 1 = 0$:

   (a) Find the discriminant in terms of k

   (b) Find the values of k for which the equation has equal roots

   (c) Find the values of k for which the equation has no real roots

   (d) When $k = 3$, find the sum and product of the roots

9. The quadratic $cx^2 + dx + e = 0$ has roots α and β.

   (a) Express $\alpha + \beta$ and $\alpha\beta$ in terms of c, d, and e

   (b) Find a quadratic equation with roots $\alpha - 5$ and $\beta - 5$

   (c) Find a quadratic equation with roots $3\alpha - 2$ and $3\beta - 2$

   (d) If $\alpha^2 + \beta^2 = 30$ and $\alpha + \beta = 9$, find $\alpha\beta$

Section C: Cubic and Higher Order Equations

10. Solve these cubic equations:

    (a) $x^3 - 10x^2 + 31x - 30 = 0$

    (b) $x^3 + 6x^2 - 7x - 60 = 0$

    (c) $9x^3 - 3x^2 - 31x - 12 = 0$

    (d) $x^3 - 18x^2 + 107x - 210 = 0$

11. Given that $x = 5$ is a root of $x^3 - 8x^2 + cx + 10 = 0$:

    (a) Find the value of c

    (b) Factor the cubic completely

    (c) Find all three roots

    (d) Verify by substitution that all roots satisfy the equation

12. Solve these quartic equations:

(a) $x^4 - 29x^2 + 100 = 0$

(b) $x^4 - 15x^2 + 54 = 0$

(c) $(x^2 + 3x)^2 - 4(x^2 + 3x) - 12 = 0$

(d) $x^4 - 5x^3 - 2x^2 + 24x - 18 = 0$ (given that $x = 3$ is a root)

13. Use the substitution $w = x^2 - \frac{1}{x^2}$ to solve:

(a) $x^4 + \frac{1}{x^4} = 7$

(b) $2x^4 - 3x^2 + \frac{6}{x^2} - \frac{4}{x^4} = 0$

Section D: Functions - Definition and Notation

14. For the function $f(x) = \frac{7x+3}{5x-4}$ where $x \neq \frac{4}{5}$:

(a) Find $f(0)$, $f(1)$, and $f(-2)$

(b) Solve $f(x) = 6$

(c) Find the value of x for which $f(x)$ is undefined

(d) Find the range of $f(x)$

15. Given $g(x) = x^2 + 16x + 55$:

(a) Express $g(x)$ in the form $(x + p)^2 + q$

(b) State the minimum value of $g(x)$ and the value of x at which it occurs

(c) Solve $g(x) = 0$

(d) Find the range of $g(x)$

16. For $h(x) = \sqrt{64 - x^2}$:

(a) Find the domain of $h(x)$

(b) Find the range of $h(x)$

(c) Sketch the graph of $y = h(x)$

(d) Solve $h(x) = 7$

17. Define $k(x) = \begin{cases} 5x^2 - 3 & \text{if } x \leq -1 \\ 4x + 1 & \text{if } -1 < x < 4 \\ 17 & \text{if } x \geq 4 \end{cases}$

(a) Find $k(-2)$, $k(-1)$, $k(3)$, and $k(5)$

(b) Is $k(x)$ continuous at $x = -1$? Justify your answer

(c) Is $k(x)$ continuous at $x = 4$? Justify your answer

(d) Sketch the graph of $y = k(x)$

Section E: Composite and Inverse Functions

18. Given $f(x) = 10x + 7$ and $g(x) = x^2 - 6$:

(a) Find $f(g(x))$ and $g(f(x))$

(b) Calculate $f(g(2))$ and $g(f(2))$

(c) Solve $f(g(x)) = 47$

(d) Find $(f \circ g)^{-1}(x)$

19. For $p(x) = \frac{6x-1}{4x+5}$ where $x \neq -\frac{5}{4}$:
    (a) Find $p^{-1}(x)$
    (b) Verify that $p(p^{-1}(x)) = x$
    (c) State the domain and range of $p^{-1}(x)$
    (d) Solve $p(x) = p^{-1}(x)$

20. Given $f(x) = 11x - 5$ and $g(x) = \frac{4}{3x+2}$ where $x \neq -\frac{2}{3}$:
    (a) Find $(f \circ g)(x)$ and state its domain
    (b) Find $(g \circ f)(x)$ and state its domain
    (c) Find $(f \circ g)^{-1}(x)$
    (d) Verify your answer by showing $(f \circ g)((f \circ g)^{-1}(x)) = x$

21. The function $h(x) = x^2 + 18x + 12$ is defined for $x \geq -9$.
    (a) Explain why the domain restriction is necessary for h^{-1} to exist
    (b) Find $h^{-1}(x)$
    (c) State the domain and range of $h^{-1}(x)$
    (d) Sketch $h(x)$ and $h^{-1}(x)$ on the same axes

Section F: Graphing Functions

22. Sketch the graphs of these functions, clearly showing key features:
    (a) $y = x^3 - 12x^2 + 45x - 50$
    (b) $y = \frac{7x+3}{5x-4}$
    (c) $y = |x^2 - 16x + 55|$
    (d) $y = \frac{x^2+36}{x^2-4}$

23. For the rational function $f(x) = \frac{x^2+6x+8}{x^2-36}$:
    (a) Find the domain of $f(x)$
    (b) Find the x and y intercepts
    (c) Identify any vertical asymptotes
    (d) Find the horizontal asymptote
    (e) Sketch the graph of $y = f(x)$

24. Analyze the function $g(x) = \frac{8x^2-32}{x^2+3x-18}$:
    (a) Factorize the numerator and denominator
    (b) Simplify $g(x)$ and state its domain
    (c) Find any asymptotes
    (d) Find the coordinates of any stationary points
    (e) Sketch the graph of $y = g(x)$

25. For the polynomial $p(x) = x^4 - 12x^3 + 36x^2$:
    (a) Factorize $p(x)$ completely
    (b) Find the roots and their multiplicities
    (c) Determine the behavior at each root
    (d) Find $p'(x)$ and locate stationary points
    (e) Sketch the graph of $y = p(x)$

Section G: Function Transformations

26. Given the function $f(x) = x^2$, describe the transformations and sketch:

    (a) $y = f(x + 6) - 2$
    (b) $y = -\frac{2}{3}f(x - 4)$
    (c) $y = f(7x) + 8$
    (d) $y = 6f(-x) - 5$

27. The graph of $y = f(x)$ has vertex at $(2, -4)$ and passes through $(0, 0)$ and $(4, 0)$. Find the vertex and two other points for:

    (a) $y = f(x) + 7$
    (b) $y = f(x - 5)$
    (c) $y = 5f(x)$
    (d) $y = f(6x)$
    (e) $y = -f(x)$
    (f) $y = f(-x)$

28. Given that $g(x) = |x + 6| - 4$:

    (a) Describe the transformations applied to $y = |x|$
    (b) State the vertex of the graph
    (c) Find the range of $g(x)$
    (d) Solve $g(x) = 5$
    (e) Sketch the graph of $y = g(x)$

29. The function $h(x) = \csc x$ is transformed to $k(x) = 5\csc(3x + \frac{\pi}{2}) + 2$.

    (a) Identify each transformation in the correct order
    (b) State the period of $k(x)$
    (c) Find the phase shift
    (d) Find the vertical shift
    (e) Find the vertical asymptotes in the interval $[0, \frac{2\pi}{3}]$
    (f) Sketch one complete cycle of $y = k(x)$

Section H: Special Functions and Applications

30. For the exponential function $f(x) = 8^{x+3} - 7$:

    (a) State the domain and range
    (b) Find the y-intercept
    (c) Find the horizontal asymptote
    (d) Solve $f(x) = 57$
    (e) Find $f^{-1}(x)$ and state its domain and range

31. For the logarithmic function $g(x) = \log_7(5x + 1) - 3$:

    (a) State the domain and range
    (b) Find the x-intercept
    (c) Find the vertical asymptote

(d) Solve $g(x) = 2$

(e) Express $g(x)$ in terms of natural logarithms

32. A function is defined as $f(x) = \frac{hx+j}{kx+l}$ where $hl - jk \neq 0$.

(a) Find the domain of $f(x)$

(b) Find $f^{-1}(x)$

(c) Show that $(f^{-1} \circ f)(x) = x$

(d) Find the condition for f to satisfy $f(f(x)) = x$ (involution property)

33. The modulus function $|x|$ can be written as: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

(a) Sketch $y = |7x - 3|$

(b) Solve $|7x - 3| = 10$

(c) Solve $|7x - 3| < 9$

(d) Find the range of values for which $|7x - 3| \geq 4$

Section I: Problem Solving and Applications

34. A parabolic satellite dish has a cross-section where the focus is 8 cm from the vertex. If the dish is 32 cm wide at the rim, find the depth.

(a) Set up a coordinate system with vertex at origin

(b) Use the standard form $x^2 = 4py$ where $p = 8$

(c) Find the depth when $x = 16$

(d) If the area of the cross-section is $A = \frac{2}{3}wd$ where w is width and d is depth, calculate the area

(e) State any assumptions made in your model

35. The efficiency E (as a percentage) of a solar panel depends on temperature T (in °C) according to: $E(T) = -0.5T^2 + 20T + 100$ for $0 \leq T \leq 50$

(a) Express $E(T)$ in completed square form

(b) Find the temperature for maximum efficiency

(c) Calculate the maximum efficiency

(d) Determine the temperatures at which efficiency is 150%

(e) Find the efficiency at 30°C

36. A suspension bridge's cable follows the path: $y(x) = 0.01x^2 - 0.8x + 20$ meters, where x is horizontal distance from one end for $0 \leq x \leq 80$

(a) Find the lowest point of the cable

(b) Calculate the minimum height of the cable

(c) Determine where the cable is 18 meters high

(d) Find the height at the bridge's center ($x = 40$)

37. A function $f(x) = \frac{x^2-49}{x^2+25}$ models a transfer function in engineering.

(a) Find the domain and range of $f(x)$

(b) Determine any asymptotes and explain their engineering significance

(c) Find the zeros of the transfer function

(d) Analyze the behavior as $x \to \pm\infty$

(e) Sketch the graph and discuss symmetry properties

38. Two functions are related by $g(x) = f(7x + 3) - 6$ where $f(x) = x^2$.

(a) Find an explicit expression for $g(x)$

(b) Describe the sequence of transformations that map f to g

(c) Find the vertex of the parabola $y = g(x)$

(d) If f has domain $[-6, 3]$, find the domain of g

(e) Solve $g(x) = f(x)$ and interpret the solutions

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

For more resources and practice materials, visit:
stepupmaths.co.uk