A Level Pure Mathematics
Practice Test 4: Differentiation

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Basic Differentiation - Polynomials

1. Differentiate these polynomial functions:
   (a) $f(x) = 2x^5 - 4x^4 + 3x^3 - 6x^2 + 8x - 5$
   (b) $g(x) = 3x^4 + \frac{2}{3}x^2 - 5x + 9$
   (c) $h(x) = (x + 2)(x - 4)$
   (d) $k(x) = (3x + 1)^2$
   (e) $p(x) = x^2(x^3 + 2)$
   (f) $q(x) = \frac{x^5-3x^3+2x}{x^2}$

2. Find $\frac{dy}{dx}$ for:
   (a) $y = 5x^{-3} + 2x^{-2} - 4$
   (b) $y = \frac{3}{x^2} - \frac{4}{\sqrt{x}} + 6\sqrt{x}$
   (c) $y = 4\sqrt{x^5} + \frac{2}{x^3} - x^{-\frac{2}{3}}$
   (d) $y = (2x - \frac{3}{x})^2$

3. Find the gradient of these curves at the given points:
   (a) $y = x^4 - 2x^3 + 5x - 1$ at $x = 1$
   (b) $y = 3x^3 - 4x^2 + 2$ at $x = -2$
   (c) $y = \frac{2x^2-1}{x}$ at $x = 2$
   (d) $y = (x + 1)^3$ at $x = 0$

4. Find the equation of the tangent line to:
   (a) $y = 2x^3 - x^2 + 3x - 2$ at the point where $x = 2$
   (b) $y = x^2 - 4x + 5$ at the point $(3, 2)$
   (c) $y = x^3 - 3x$ at the point where the gradient is 9
   (d) $y = \frac{x^3}{3} - 2x + 4$ at the point where $x = 3$

5. Given that $f(x) = px^3 + qx^2 + rx + s$ and $f'(x) = 9x^2 - 18x + 6$:
   (a) Find the values of p, q, and r
   (b) If $f(1) = 8$, find the value of s
   (c) Write the complete expression for $f(x)$
   (d) Find $f(3)$ and $f'(0)$

Section B: Differentiation of Special Functions

6. Differentiate these exponential and logarithmic functions:
   (a) $f(x) = 2e^x$
   (b) $g(x) = 4e^x + 3x^3$
   (c) $h(x) = x^2e^x$
   (d) $k(x) = 3\ln x$
   (e) $p(x) = x^2 \ln x$
   (f) $q(x) = \frac{e^x}{x}$
7. Differentiate these trigonometric functions:
   (a) $f(x) = 2\sin x + 3\cos x$
   (b) $g(x) = 4\sin x - \cos x + 2x^3$
   (c) $h(x) = x^2 \sin x$
   (d) $k(x) = \frac{\sin x}{x}$
   (e) $p(x) = 2\tan x$
   (f) $q(x) = x$
8. Find $\frac{dy}{dx}$ for:
   (a) $y = e^{3x}$
   (b) $y = \ln(2x)$
   (c) $y = \sin(3x)$
   (d) $y = \cos(3x + 2)$
   (e) $y = e^{2x^2}$
   (f) $y = \ln(x^3 - 2)$
9. Differentiate using appropriate rules:
   (a) $f(x) = e^x \sin x$
   (b) $g(x) = x^3 \cos x$
   (c) $h(x) = \frac{\ln x}{x}$
   (d) $k(x) = \frac{\cos x}{\sin x}$
   (e) $p(x) = (\ln x)^3$
   (f) $q(x) = \sqrt{\cos x}$
10. Find the derivatives of:
    (a) $f(x) = \cos^2 x$
    (b) $g(x) = \sin^4 x$
    (c) $h(x) = e^{\cos x}$
    (d) $k(x) = \ln(\sin x)$
    (e) $p(x) = (\sin x - \cos x)^2$
    (f) $q(x) = \sin^{-1} x$ (inverse sin)

Section C: Product Rule and Quotient Rule

11. Use the product rule to differentiate:
    (a) $f(x) = (x^3 + 2)(x^2 - 1)$
    (b) $g(x) = (3x + 1)(x^3 - 2x + 3)$
    (c) $h(x) = x^3 e^x$
    (d) $k(x) = (x - 2)\ln x$
    (e) $p(x) = \sin x \sin x$
    (f) $q(x) = x^2 \cos x$

12. Use the quotient rule to differentiate:
    (a) $f(x) = \frac{x^3-1}{x+2}$
    (b) $g(x) = \frac{3x-1}{x^2+2}$
    (c) $h(x) = \frac{e^x}{x^3}$
    (d) $k(x) = \frac{\ln x}{x-2}$
    (e) $p(x) = \frac{\cos x}{1+\sin x}$
    (f) $q(x) = \frac{x^3}{\cos x}$

13. Choose the most appropriate method to differentiate:
    (a) $f(x) = \frac{x^4-3x^2}{x^2}$
    (b) $g(x) = (x^3 + 1)(x - 3)$
    (c) $h(x) = \frac{x^3-2x+5}{x^3}$
    (d) $k(x) = x^2(x^2 - 1)^3$
    (e) $p(x) = \frac{(x-1)^3}{x^2}$
    (f) $q(x) = x^3\sqrt{x - 2}$

14. Given $f(x) = x^3$ and $g(x) = \cos x$:
    (a) Find $(fg)'(x)$ using the product rule
    (b) Find $(\frac{f}{g})'(x)$ using the quotient rule
    (c) Evaluate $(fg)'(\frac{\pi}{3})$
    (d) Evaluate $(\frac{f}{g})'(\frac{\pi}{4})$

15. Prove these differentiation rules:
    (a) Product rule: $(uv)' = u'v + uv'$
    (b) Quotient rule: $(\frac{u}{v})' = \frac{u'v-uv'}{v^2}$
    (c) Show that $(\frac{1}{v})' = -\frac{v'}{v^2}$
    (d) Verify that $(uvw)' = u'vw + uv'w + uvw'$

Section D: Chain Rule

16. Use the chain rule to differentiate:
    (a) $f(x) = (3x-2)^4$
    (b) $g(x) = (x^3+2x-1)^5$
    (c) $h(x) = \sqrt{2x^2-3}$
    (d) $k(x) = (4x+1)^{-3}$
    (e) $p(x) = \sin(3x-1)$
    (f) $q(x) = \cos(x^3)$

17. Find $\frac{dy}{dx}$ for:
    (a) $y = e^{2x+3}$
    (b) $y = \ln(4x-1)$
    (c) $y = (x^3-2x)^6$
    (d) $y = \cos^2 x$
    (e) $y = \sin(e^x)$
    (f) $y = e^{\cos x}$

18. Differentiate these composite functions:
    (a) $f(x) = (e^x-1)^4$
    (b) $g(x) = \ln(x^3+3x-1)$
    (c) $h(x) = \cos(\ln x)$
    (d) $k(x) = e^{x \sin x}$
    (e) $p(x) = (\sin x + \cos x)^3$
    (f) $q(x) = \ln(\cos x)$

19. Use multiple rules to differentiate:
    (a) $f(x) = x^2(3x-1)^4$
    (b) $g(x) = \frac{x^3}{(x-1)^3}$
    (c) $h(x) = x^3 \sin(2x)$
    (d) $k(x) = e^x \cos(3x)$
    (e) $p(x) = \frac{\ln x}{\sqrt{x^2+1}}$
    (f) $q(x) = \frac{(x^3-1)^2}{x^2}$

20. Find the second derivatives:
    (a) $f(x) = (x-2)^5$
    (b) $g(x) = \sin(3x)$
    (c) $h(x) = e^{2x}$
    (d) $k(x) = \ln(x^3)$
    (e) $p(x) = x^3 e^x$
    (f) $q(x) = \sin x \sin x$

Section E: Stationary Points

21. Find the coordinates of stationary points for:
    (a) $f(x) = x^3 - 6x^2 + 9x + 1$
    (b) $g(x) = 3x^3 - 9x^2 + 6x + 2$
    (c) $h(x) = x^4 - 8x^2 + 12$
    (d) $k(x) = \frac{x^3}{x+1}$ for $x \neq -1$
22. Determine the nature of each stationary point using the second derivative test:
    (a) $f(x) = x^3 - 3x^2 + 3x - 2$
    (b) $g(x) = 3x^3 - 6x^2 - 15x + 1$
    (c) $h(x) = x^4 - 4x^2 + 5$
    (d) $k(x) = x^2e^{-x}$
23. Find and classify all stationary points:
    (a) $f(x) = x^3 + 3x^2 - 9x + 5$
    (b) $g(x) = 3x^3 - 4x^2 - 18x + 2$
    (c) $h(x) = x^4 - 12x^2 + 36$
    (d) $k(x) = x - \frac{4}{x}$ for $x > 0$
24. For the function $f(x) = bx^3 + cx^2 + dx + e$:
    (a) Find the conditions on b, c, and d for the function to have two stationary points
    (b) If $f(x) = 2x^3 - 6x^2 + 6x - 1$, show it has no stationary points
    (c) Find the values of m for which $f(x) = x^3 - 6mx + 3$ has exactly one stationary point
25. Analyze the function $f(x) = \frac{x^3-9}{x}$:
    (a) Find the domain of $f(x)$
    (b) Find $f'(x)$ and locate stationary points
    (c) Determine the nature of stationary points
    (d) Find any asymptotes
    (e) Sketch the graph of $y = f(x)$

Section F: Rates of Change

26. A particle moves along a line with position $s(t) = 2t^3 - 9t^2 + 12t + 5$ meters at time t seconds.
    (a) Find the velocity $v(t)$ and acceleration $a(t)$
    (b) Find when the particle is at rest
    (c) Calculate the velocity and acceleration at $t = 3$
    (d) Determine when the acceleration is zero
    (e) Find the displacement between $t = 0$ and $t = 3$
27. The volume of a cube is $V = s^3$. If the side length increases at a rate of 3 cm/s:
    (a) Find the rate of change of volume when $s = 4$ cm
    (b) Express $\frac{dV}{dt}$ in terms of s and $\frac{ds}{dt}$
    (c) When is the volume increasing at 300 cm^3/s?

(d) Find the rate of change of surface area when $s = 6$ cm

28. A ladder 8 meters long leans against a vertical wall. The bottom slides away at 2 m/s.
    (a) Set up the relationship between distances
    (b) Find how fast the top slides down when the bottom is 6m from the wall
    (c) Find the rate of change of the angle with the ground
    (d) When is the top sliding down fastest?

29. Water flows into a cylindrical tank at 3 m^3/min. The tank has radius 2m.
    (a) Express the volume in terms of height h
    (b) Find how fast the water level rises
    (c) Find the rate of change of surface area when $h = 3$m
    (d) How long to fill a 5m tall tank?

30. The population of a bacteria culture grows according to $P(t) = 5000e^{0.03t}$ where t is hours.
    (a) Find the growth rate $\frac{dP}{dt}$
    (b) Calculate the population and growth rate after 4 hours
    (c) When is the population growing at 200 bacteria per hour?
    (d) Express the growth rate as a percentage of current population

Section G: Optimization Problems

31. A farmer has 300m of fencing to enclose a rectangular field with a divider down the middle.
    (a) Express the area in terms of one variable
    (b) Find the dimensions for maximum area
    (c) Calculate the maximum area
    (d) Verify this is a maximum using the second derivative

32. A cylindrical container with a lid has volume 50 m^3. The material for the base and lid costs £8/m^2, sides cost £4/m^2.
    (a) Express the cost in terms of the radius
    (b) Find dimensions for minimum cost
    (c) Calculate the minimum cost
    (d) Find the ratio of height to radius

33. A company's revenue function is $R(x) = -x^3 + 15x^2 + 60x - 50$ thousand pounds, where x is advertising spend (thousands).
    (a) Find the advertising levels for maximum and minimum revenue
    (b) Calculate the maximum revenue
    (c) Find the marginal revenue function
    (d) Determine the optimal advertising spend

34. A poster has area 300 cm^2. It has margins of 2cm on all sides around the printed area.
    (a) Express the printed area in terms of poster width
    (b) Find dimensions for maximum printed area

(c) Calculate the maximum printed area

(d) Find the ratio of poster height to width

35. A rectangular package must have length + girth 108 cm (girth = perimeter of cross-section).

 (a) Express volume in terms of cross-section dimensions
 (b) Find dimensions for maximum volume
 (c) Calculate the maximum volume
 (d) Verify this gives a maximum

Section H: Implicit Differentiation and Related Rates

36. Find $\frac{dy}{dx}$ using implicit differentiation:

 (a) $x^2 + y^2 = 36$
 (b) $x^2 - 3xy + y^2 = 9$
 (c) $x^3 - y^3 = 8xy$
 (d) $\cos(xy) = x - y$
 (e) $e^{x+y} = xy$
 (f) $\ln(x + y) = x + y$

37. Find the equation of the tangent to these curves at the given points:

 (a) $x^2 + y^2 = 25$ at $(3, 4)$
 (b) $x^2 + xy + y^2 = 19$ at $(2, 3)$
 (c) $x^3 - y^3 = 7$ at $(2, 1)$
 (d) $ye^x = 3$ at $(0, 3)$

38. Use implicit differentiation to find $\frac{d^2y}{dx^2}$:

 (a) $x^2 - y^2 = 9$
 (b) $xy = 4$
 (c) $x^2 + y^2 = 16$

39. Two ships start from ports 50 km apart. Ship A travels south at 30 km/h, Ship B travels west at 40 km/h.

 (a) Express the distance between ships as a function of time
 (b) Find how fast they're separating after 1 hour
 (c) When are they separating at 60 km/h?
 (d) Find the minimum distance between them

40. A spherical balloon is deflated so its volume decreases at 80 cm^3/s. Find the rate of decrease of:

 (a) Radius when $r = 8$ cm
 (b) Surface area when $r = 6$ cm
 (c) Diameter when volume is 2000 cm^3
 (d) The rate when surface area is 300π cm^2

Section I: Advanced Applications

41. A Gothic window has the shape of a rectangle topped by an equilateral triangle, with total perimeter 24m.
    (a) Find dimensions to maximize the area
    (b) Calculate the maximum area
    (c) Find the optimal ratio of rectangle height to width
    (d) Determine what fraction of area is rectangular
42. The stiffness of a rectangular beam is proportional to w^3d where w is width and d is depth. A beam is cut from a circular log of radius 15 cm.
    (a) Express stiffness in terms of width w
    (b) Find dimensions for maximum stiffness
    (c) Calculate the ratio $\frac{w}{d}$ for stiffest beam
    (d) Compare with beam of square cross-section
43. A medication concentration in blood follows $C(t) = \frac{Bt^2}{(t+2)^3}$ mg/L where t is hours after injection.
    (a) Find when concentration is maximum
    (b) If peak concentration is 3 mg/L, find B
    (c) Calculate the rate of change at $t = 2$
    (d) Find when concentration is decreasing fastest
    (e) Determine the half-life from peak concentration
44. An isosceles triangle is inscribed in a circle of radius 10 cm.
    (a) Express the area in terms of the base angle
    (b) Find the angle for maximum area
    (c) Calculate the maximum area
    (d) Show the optimal triangle is equilateral
45. A workshop's hourly profit is $P(x) = 0.2x^3 - 2.4x^2 + 8x + 20$ for employing x workers.
    (a) Find the marginal profit function
    (b) Determine the number of workers for maximum profit
    (c) Calculate the maximum hourly profit
    (d) Find when marginal profit equals zero
    (e) Graph the profit function and interpret economically
46. Two positive numbers have product 48. Find the numbers that:
    (a) Minimize their sum
    (b) Maximize the sum of their squares
    (c) Minimize the sum of their cube roots
    (d) Maximize $x^3 + y^2$ where $xy = 48$
    (e) Explain why the answers differ
47. A satellite follows an elliptical orbit with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
    (a) Find the slope at any point (x, y) on the orbit

(b) Find points where the tangent is horizontal
(c) Calculate the slope at the point $(a\cos\theta, b\sin\theta)$
(d) Find the angle between position vector and tangent
(e) Derive conditions for circular orbit

48. Design an optimization problem from your own experience:

(a) Define your scenario and variables clearly
(b) Set up the objective function and constraints
(c) Use differentiation to find optimal solutions
(d) Verify your solution makes practical sense
(e) Discuss real-world factors not in your model

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

For more resources and practice materials, visit:
stepupmaths.co.uk