A Level Pure Mathematics
Practice Test 5: Algebra and Functions

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Algebraic Manipulation

1. Simplify these expressions:
   (a) $\frac{x^2-49}{x^2+14x+49}$
   (b) $\frac{6x^2-24}{x^2-2x-8}$
   (c) $\frac{x^3-216}{x^2-36}$
   (d) $\frac{x^4-1296}{x^2+6x-72}$
2. Factorize completely:
   (a) $x^3 + 15x^2 + 75x + 125$
   (b) $216x^3 - 343$
   (c) $x^8 - 256$
   (d) $x^{15} - 1$
   (e) $x^4 + 18x^2 + 81$
   (f) $x^3 - 6x^2 + 11x - 30$
3. Express as single fractions in simplest form:
   (a) $\frac{6}{x-3} + \frac{4}{x+5}$
   (b) $\frac{5x}{x^2-25} - \frac{2}{x+5}$
   (c) $\frac{3x+1}{x^2-4x+3} + \frac{x-5}{x^2+x-20}$
   (d) $\frac{5}{x+3} - \frac{4}{x-1} + \frac{6}{x^2+2x-3}$
4. Use the binomial theorem to expand:
   (a) $(6x - 1)^4$
   (b) $(2x + \frac{3}{2x})^6$
   (c) $(4 - x)^8$, and find the coefficient of x^6
   (d) Find the term containing x^3 in the expansion of $(x^5 - \frac{2}{x^3})^9$
5. Simplify using laws of indices:
   (a) $\frac{11^{2x+1} \cdot 121^{x-3}}{1331^{x-1}}$

(b) $\frac{243^{x+2} \cdot 81^{2x}}{729^{x+1}}$
(c) $(x^{\frac{5}{6}})^{\frac{6}{7}} \cdot x^{-\frac{4}{5}}$
(d) $\frac{(8x)^2 \cdot (3x^4)^2}{24x^9}$

Section B: Linear and Quadratic Equations

6. Solve these equations:
   (a) $\frac{6x-5}{7} - \frac{4x+1}{3} = \frac{2}{21}$
   (b) $\frac{5x}{x+2} = \frac{7}{x-4}$
   (c) $\sqrt{6x-5} = 4x - 7$
   (d) $\frac{5}{x-3} - \frac{3}{x+4} = \frac{2}{7}$
7. Solve these quadratic equations, leaving answers in exact form where appropriate:
   (a) $7x^2 - 13x + 4 = 0$
   (b) $x^2 - 12x + 11 = 0$
   (c) $6x^2 = 11x + 2$
   (d) $(6x-5)^2 = 4(2x+3)$
8. For the quadratic equation $5x^2 + (4k-3)x + 3k - 1 = 0$:
   (a) Find the discriminant in terms of k
   (b) Find the values of k for which the equation has equal roots
   (c) Find the values of k for which the equation has no real roots
   (d) When $k = 2$, find the sum and product of the roots
9. The quadratic $ux^2 + vx + w = 0$ has roots α and β.
   (a) Express $\alpha + \beta$ and $\alpha\beta$ in terms of u, v, and w
   (b) Find a quadratic equation with roots $\alpha + 4$ and $\beta + 4$
   (c) Find a quadratic equation with roots $2\alpha + 1$ and $2\beta + 1$
   (d) If $\alpha^2 + \beta^2 = 26$ and $\alpha + \beta = 8$, find $\alpha\beta$

Section C: Cubic and Higher Order Equations

10. Solve these cubic equations:
    (a) $x^3 - 9x^2 + 26x - 24 = 0$
    (b) $x^3 + 5x^2 - 2x - 24 = 0$
    (c) $8x^3 - 6x^2 - 23x + 15 = 0$
    (d) $x^3 - 15x^2 + 74x - 120 = 0$
11. Given that $x = 4$ is a root of $x^3 - 7x^2 + bx - 20 = 0$:
    (a) Find the value of b
    (b) If one of the other roots is 1, find the third root
    (c) Write the equation in fully factored form
    (d) Verify by expanding your factored form
12. Solve these quartic equations:

(a) $x^4 - 25x^2 + 144 = 0$
(b) $x^4 - 14x^2 + 45 = 0$
(c) $(x^2 - 4x)^2 - 3(x^2 - 4x) - 10 = 0$
(d) $x^4 - 4x^3 - 7x^2 + 22x + 24 = 0$ (given that $x = -1$ is a root)

13. Use the substitution $z = x + \frac{4}{x}$ to solve:

(a) $x^2 + \frac{16}{x^2} = 17$
(b) $3x^2 - 5x + \frac{20}{x} - \frac{12}{x^2} = 0$

Section D: Functions - Definition and Notation

14. For the function $f(x) = \frac{6x-5}{4x+3}$ where $x \neq -\frac{3}{4}$:

(a) Find $f(0)$, $f(2)$, and $f(-1)$
(b) Solve $f(x) = 5$
(c) Find the value of x for which $f(x)$ is undefined
(d) Find the range of $f(x)$

15. Given $g(x) = x^2 - 14x + 45$:

(a) Express $g(x)$ in the form $(x - m)^2 + n$
(b) State the minimum value of $g(x)$ and the value of x at which it occurs
(c) Solve $g(x) = 0$
(d) Find the range of $g(x)$

16. For $h(x) = \sqrt{49 - x^2}$:

(a) Find the domain of $h(x)$
(b) Find the range of $h(x)$
(c) Sketch the graph of $y = h(x)$
(d) Solve $h(x) = 6$

17. Define $k(x) = \begin{cases} 4x^2 + 2 & \text{if } x < -2 \\ 3x - 4 & \text{if } -2 \leq x \leq 3 \\ 5 & \text{if } x > 3 \end{cases}$

(a) Find $k(-3)$, $k(-2)$, $k(1)$, and $k(4)$
(b) Is $k(x)$ continuous at $x = -2$? Justify your answer
(c) Is $k(x)$ continuous at $x = 3$? Justify your answer
(d) Sketch the graph of $y = k(x)$

Section E: Composite and Inverse Functions

18. Given $f(x) = 8x - 3$ and $g(x) = x^2 + 5$:

(a) Find $f(g(x))$ and $g(f(x))$
(b) Calculate $f(g(-1))$ and $g(f(-1))$
(c) Solve $f(g(x)) = 37$
(d) Find $(f \circ g)^{-1}(x)$

19. For $p(x) = \frac{5x+2}{3x-4}$ where $x \neq \frac{4}{3}$:

    (a) Find $p^{-1}(x)$
    (b) Verify that $p(p^{-1}(x)) = x$
    (c) State the domain and range of $p^{-1}(x)$
    (d) Solve $p(x) = p^{-1}(x)$

20. Given $f(x) = 9x + 4$ and $g(x) = \frac{1}{2x-3}$ where $x \neq \frac{3}{2}$:

    (a) Find $(f \circ g)(x)$ and state its domain
    (b) Find $(g \circ f)(x)$ and state its domain
    (c) Find $(f \circ g)^{-1}(x)$
    (d) Verify your answer by showing $(f \circ g)((f \circ g)^{-1}(x)) = x$

21. The function $h(x) = x^2 + 16x + 7$ is defined for $x \geq -8$.

    (a) Explain why the domain restriction is necessary for h^{-1} to exist
    (b) Find $h^{-1}(x)$
    (c) State the domain and range of $h^{-1}(x)$
    (d) Sketch $h(x)$ and $h^{-1}(x)$ on the same axes

Section F: Graphing Functions

22. Sketch the graphs of these functions, clearly showing key features:

    (a) $y = x^3 - 9x^2 + 24x - 16$
    (b) $y = \frac{6x-5}{4x+3}$
    (c) $y = |x^2 - 14x + 45|$
    (d) $y = \frac{x^2+25}{x^2-9}$

23. For the rational function $f(x) = \frac{x^2-5x+6}{x^2-25}$:

    (a) Find the domain of $f(x)$
    (b) Find the x and y intercepts
    (c) Identify any vertical asymptotes
    (d) Find the horizontal asymptote
    (e) Sketch the graph of $y = f(x)$

24. Analyze the function $g(x) = \frac{7x^2-28}{x^2-4x-12}$:

    (a) Factorize the numerator and denominator
    (b) Simplify $g(x)$ and state its domain
    (c) Find any asymptotes
    (d) Find the coordinates of any stationary points
    (e) Sketch the graph of $y = g(x)$

25. For the polynomial $p(x) = x^4 - 10x^3 + 25x^2$:

    (a) Factorize $p(x)$ completely
    (b) Find the roots and their multiplicities
    (c) Determine the behavior at each root
    (d) Find $p'(x)$ and locate stationary points
    (e) Sketch the graph of $y = p(x)$

Section G: Function Transformations

26. Given the function $f(x) = x^2$, describe the transformations and sketch:
    (a) $y = f(x-5) + 4$
    (b) $y = -\frac{3}{2}f(x+3)$
    (c) $y = f(6x) - 7$
    (d) $y = 5f(-x) + 3$

27. The graph of $y = f(x)$ has vertex at $(-1, 5)$ and passes through $(1, 9)$ and $(-3, 9)$. Find the vertex and two other points for:
    (a) $y = f(x) + 6$
    (b) $y = f(x-4)$
    (c) $y = 3f(x)$
    (d) $y = f(5x)$
    (e) $y = -f(x)$
    (f) $y = f(-x)$

28. Given that $g(x) = |x-5| + 3$:
    (a) Describe the transformations applied to $y = |x|$
    (b) State the vertex of the graph
    (c) Find the range of $g(x)$
    (d) Solve $g(x) = 8$
    (e) Sketch the graph of $y = g(x)$

29. The function $h(x) = \sec x$ is transformed to $k(x) = 4\sec(2x - \frac{\pi}{3}) - 1$.
    (a) Identify each transformation in the correct order
    (b) State the period of $k(x)$
    (c) Find the phase shift
    (d) Find the vertical shift
    (e) Find the vertical asymptotes in the interval $[0, 2\pi]$
    (f) Sketch one complete cycle of $y = k(x)$

Section H: Special Functions and Applications

30. For the exponential function $f(x) = 6^{x-2} + 3$:
    (a) State the domain and range
    (b) Find the y-intercept
    (c) Find the horizontal asymptote
    (d) Solve $f(x) = 39$
    (e) Find $f^{-1}(x)$ and state its domain and range

31. For the logarithmic function $g(x) = \log_6(4x-3) + 1$:
    (a) State the domain and range
    (b) Find the x-intercept
    (c) Find the vertical asymptote

(d) Solve $g(x) = 4$

(e) Express $g(x)$ in terms of natural logarithms

32. A function is defined as $f(x) = \frac{dx+e}{fx+g}$ where $dg - ef \neq 0$.

(a) Find the domain of $f(x)$

(b) Find $f^{-1}(x)$

(c) Show that $(f^{-1} \circ f)(x) = x$

(d) Determine when $f(x) = -x$ and interpret geometrically

33. The modulus function $|x|$ can be written as: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

(a) Sketch $y = |6x + 1|$

(b) Solve $|6x + 1| = 11$

(c) Solve $|6x + 1| \leq 8$

(d) Find the range of values for which $|6x + 1| > 5$

Section I: Problem Solving and Applications

34. A Norman window consists of a rectangle surmounted by a semicircle. The perimeter is 24 meters. Let x be the width of the rectangle.

(a) Express the height of the rectangle in terms of x

(b) Show that the area $A = x(12 - \frac{x}{2} - \frac{\pi x}{4})$

(c) Find the value of x that maximizes the area

(d) Calculate the maximum area

(e) State the domain of the function in this context

35. The temperature T (in degrees Celsius) of a cooling object after t minutes is given by: $T(t) = 3t^2 - 36t + 120$

(a) Express $T(t)$ in completed square form

(b) Find when the object reaches its minimum temperature

(c) Calculate the minimum temperature

(d) Determine when the object returns to 90°C

(e) Find the temperature after 8 minutes

36. A population of bacteria grows according to the function: $P(t) = -t^2 + 14t + 40$ thousand, where t is time in hours for $0 \leq t \leq 20$

(a) Find when the population reaches its maximum

(b) Calculate the maximum population

(c) Determine when the population drops to 75 thousand

(d) Find the population after 10 hours

37. A function $f(x) = \frac{x^2-36}{x^2+16}$ models a filter response.

(a) Find the domain and range of $f(x)$

(b) Determine any asymptotes and explain their significance

(c) Find when $f(x) = 0$

(d) Analyze the behavior as $x \to \pm\infty$

(e) Sketch the graph and identify any symmetry

38. Two functions are related by $g(x) = f(6x - 2) + 5$ where $f(x) = x^2$.

(a) Find an explicit expression for $g(x)$

(b) Describe the transformations that map f to g

(c) Find the vertex of the parabola $y = g(x)$

(d) If f has domain $[-5, 2]$, find the domain of g

(e) Solve $g(x) = f(x)$ and interpret the solutions geometrically

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

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