

A Level Pure Mathematics

Practice Test 1: Numerical Methods

Instructions:

Answer all questions. Show your working clearly.

Calculators may be used unless stated otherwise.

Time allowed: 2 hours

Section A: Introduction to Numerical Methods

1. Explain why numerical methods are needed for the following equations:

(a) $x^3 + 2x - 5 = 0$

(b) $x = \cos x$

(c) $e^x = 3x$

(d) $x^5 - 2x^3 + x - 1 = 0$

2. For each function, determine the approximate location of roots by examining sign changes:

(a) $f(x) = x^3 - 2x - 1$ for $x \in [-2, 2]$

(b) $f(x) = x^2 - 3x - 2$ for $x \in [-2, 5]$

(c) $f(x) = e^x - 2x - 1$ for $x \in [-1, 2]$

(d) $f(x) = \ln x - x + 2$ for $x \in [1, 4]$

3. State the conditions required for the Intermediate Value Theorem and explain how it guarantees the existence of roots.

4. For the function $f(x) = x^3 - x - 1$:

(a) Show that there is a root between $x = 1$ and $x = 2$

(b) Determine a more precise interval containing the root

(c) Sketch the graph of $y = f(x)$ showing the root location

(d) Explain why this equation cannot be solved algebraically

5. Define the following terms in the context of numerical methods:

(a) Absolute error

(b) Relative error

(c) Tolerance

(d) Convergence

(e) Iteration

(f) Fixed point

Section B: Bisection Method

6. Use the bisection method to find the root of $f(x) = x^3 - 2x - 5$ in the interval $[2, 3]$.
 - (a) Complete 4 iterations
 - (b) Give your answer correct to 2 decimal places
 - (c) Estimate the error in your final approximation
 - (d) How many iterations would be needed for an accuracy of 10^{-6} ?
7. Apply the bisection method to solve $x = \cos x$:
 - (a) Show that a root lies between 0 and 1
 - (b) Perform 5 iterations starting with $[0, 1]$
 - (c) Give your answer to 4 decimal places
 - (d) Verify your answer by substitution
8. Use the bisection method to find the positive root of $x^2 - 3x - 1 = 0$:
 - (a) Determine a suitable starting interval
 - (b) Perform iterations until the root is accurate to 3 decimal places
 - (c) Compare with the exact solution using the quadratic formula
 - (d) Calculate the absolute error
9. For the equation $e^x = 3x$:
 - (a) Show graphically that there are two roots
 - (b) Use bisection to find the smaller positive root to 3 decimal places
 - (c) Find the larger root to 3 decimal places
 - (d) Discuss the convergence rate of the bisection method
10. The equation $\ln x = 2 - x$ has a root near $x = 1.5$.
 - (a) Use bisection method with initial interval $[1, 2]$
 - (b) Continue until consecutive approximations differ by less than 0.01
 - (c) How many iterations were required?
 - (d) What is the theoretical minimum number of iterations needed?

Section C: Newton-Raphson Method

11. Use the Newton-Raphson method to solve $x^3 - 2x - 5 = 0$ starting with $x_0 = 2$.
 - (a) Write down the iteration formula
 - (b) Perform 4 iterations
 - (c) Give your answer to 5 decimal places
 - (d) Compare the convergence with bisection method
12. Apply Newton-Raphson to find $\sqrt{7}$ by solving $x^2 - 7 = 0$:
 - (a) Derive the iteration formula
 - (b) Start with $x_0 = 2$ and perform 4 iterations
 - (c) Compare with the exact value
 - (d) Explain why this converges so quickly

13. Solve $\cos x = x$ using Newton-Raphson method:
- (a) Rearrange to standard form $f(x) = 0$
 - (b) Find $f'(x)$ and write the iteration formula
 - (c) Use $x_0 = 0.5$ and perform 5 iterations
 - (d) Check your answer by substitution
14. Use Newton-Raphson to solve $e^x - 2x - 1 = 0$:
- (a) Find the iteration formula
 - (b) Starting with $x_0 = 1$, find the root to 6 decimal places
 - (c) Starting with $x_0 = -1$, find the other root
 - (d) Discuss the importance of choosing good initial values
15. Investigate the convergence of Newton-Raphson for $f(x) = x^3 - 3x + 1$:
- (a) Find all roots using different starting values
 - (b) Identify cases where the method fails to converge
 - (c) Explain why some starting values lead to divergence
 - (d) Sketch the function and its derivative to illustrate your findings
16. For the equation $x^4 - 5x^2 + 4 = 0$:
- (a) Solve exactly by substitution
 - (b) Use Newton-Raphson to find all four roots
 - (c) Compare numerical and exact solutions
 - (d) Discuss which starting values work best for each root

Section D: Fixed Point Iteration

17. Rearrange $x^2 - 3x + 1 = 0$ into the form $x = g(x)$ in different ways:
- (a) $x = \frac{x^2+1}{3}$
 - (b) $x = 3 - \frac{1}{x}$
 - (c) $x = \sqrt{3x-1}$
 - (d) Test each rearrangement for convergence near $x = 2.618$
18. Use fixed point iteration to solve $x = \cos x$:
- (a) Use the iteration $x_{n+1} = \cos x_n$ with $x_0 = 0.5$
 - (b) Perform 10 iterations
 - (c) Plot the values to show convergence
 - (d) Explain why this method converges
19. Solve $x^3 - x - 1 = 0$ using fixed point iteration:
- (a) Try the rearrangement $x = x^3 - 1$
 - (b) Try the rearrangement $x = \sqrt[3]{x+1}$
 - (c) Try the rearrangement $x = \frac{1}{x^2-1}$
 - (d) Determine which rearrangements converge and why
20. For the equation $e^x = 2x$:

- (a) Show that $x = \frac{e^x}{2}$ diverges from $x_0 = 1$
 - (b) Try $x = \ln(2x)$ starting from $x_0 = 1$
 - (c) Explain the convergence behavior using $|g'(x)|$
 - (d) Find the root to 4 decimal places
21. Investigate the convergence condition $|g'(x)| < 1$:
- (a) For $g(x) = \frac{x^2+1}{3}$, find $g'(x)$ and determine convergence regions
 - (b) For $g(x) = \sqrt{3x-1}$, analyze convergence
 - (c) Explain why some iterations converge while others diverge
 - (d) Relate this to the graphical interpretation of fixed point iteration

Section E: Trapezium Rule

22. Use the trapezium rule with 4 strips to approximate:

- (a) $\int_0^2 x^2 dx$
- (b) $\int_1^3 \frac{1}{x} dx$
- (c) $\int_0^1 e^x dx$
- (d) $\int_0^{\pi/2} \sin x dx$

Compare with exact values and calculate absolute errors.

23. Apply the trapezium rule to $\int_0^1 \sqrt{1+x^2} dx$:
- (a) Use 2 strips, then 4 strips, then 8 strips
 - (b) Comment on how the approximation improves
 - (c) Estimate the true value of the integral
 - (d) Explain why exact integration is difficult
24. For $\int_1^2 \frac{1}{x^2} dx$:
- (a) Calculate the exact value
 - (b) Use trapezium rule with $n = 2, 4, 8$ strips
 - (c) Calculate the error for each approximation
 - (d) Show that halving the strip width approximately quarters the error
25. Use the trapezium rule to estimate $\int_0^1 e^{-x^2} dx$:
- (a) Use 5 ordinates (4 strips)
 - (b) Use 9 ordinates (8 strips)
 - (c) Compare your answers and estimate the accuracy
 - (d) This integral cannot be expressed in elementary functions - explain why numerical methods are essential
26. A curve passes through points $(0, 1)$, $(0.5, 1.2)$, $(1, 1.8)$, $(1.5, 2.1)$, $(2, 2.3)$:
- (a) Use trapezium rule to find the area under the curve
 - (b) If the y -values represent velocity in m/s and x represents time in seconds, interpret your answer
 - (c) How could you improve the accuracy?
 - (d) Discuss the limitations when working with discrete data points

Section F: Simpson's Rule

27. Use Simpson's rule with 4 strips to approximate:

- (a) $\int_0^2 x^3 dx$
- (b) $\int_1^3 \frac{1}{x^2} dx$
- (c) $\int_0^1 e^{2x} dx$
- (d) $\int_0^\pi \sin x dx$

Compare with exact values and trapezium rule approximations.

28. Apply Simpson's rule to $\int_0^1 \frac{1}{1+x^2} dx$:

- (a) Use 2 strips, then 4 strips, then 8 strips
- (b) The exact value is $\frac{\pi}{4}$ - calculate errors
- (c) Compare convergence with trapezium rule
- (d) Explain why Simpson's rule is more accurate

29. For $\int_0^2 \sqrt{4-x^2} dx$:

- (a) Recognize this as the area of a semicircle
- (b) Use Simpson's rule with 4 and 8 strips
- (c) Compare with the exact value π
- (d) Calculate percentage errors

30. Use Simpson's rule to estimate $\int_1^3 \ln x dx$:

- (a) Use 4 strips
- (b) Use 8 strips
- (c) The exact value is $3 \ln 3 - 2$ - verify this and calculate errors
- (d) Discuss the convergence rate

31. A reservoir has the following cross-sectional areas at 10m intervals:

Distance (m)	0	10	20	30	40	50
Area (m ²)	120	150	180	160	140	100

- (a) Use Simpson's rule to estimate the volume of water
- (b) If water flows at 2 m/s, estimate the time to drain the reservoir
- (c) Discuss the accuracy of your approximation
- (d) What additional data would improve the estimate?

Section G: Error Analysis and Comparison of Methods

32. For $\int_0^1 x^4 dx$:

- (a) Calculate the exact value
- (b) Use trapezium rule with $n = 2, 4, 8$ strips
- (c) Use Simpson's rule with $n = 2, 4, 8$ strips
- (d) Create a table comparing errors
- (e) Verify the theoretical error formulas

33. Analyze the errors in numerical integration:
- (a) Explain why trapezium rule has error proportional to h^2
 - (b) Explain why Simpson's rule has error proportional to h^4
 - (c) For what types of functions is each method most accurate?
 - (d) Give examples where each method might be preferred
34. Compare root-finding methods for $f(x) = x^3 - 2x - 5$:
- (a) Use bisection method (6 iterations from $[2, 3]$)
 - (b) Use Newton-Raphson (4 iterations from $x_0 = 2$)
 - (c) Use fixed point iteration with $x = \sqrt[3]{2x + 5}$ (6 iterations from $x_0 = 2$)
 - (d) Compare convergence rates and accuracy
 - (e) Discuss advantages and disadvantages of each method
35. For the equation $\tan x = x$ in $(0, \pi/2)$:
- (a) Explain why bisection method works reliably
 - (b) Discuss potential problems with Newton-Raphson method
 - (c) Suggest appropriate starting values and intervals
 - (d) Find the root using your preferred method
36. Error propagation in numerical methods:
- (a) If $f(2) = 0.1$ and $f(3) = -0.2$, estimate the error in the root found by linear interpolation
 - (b) For Newton-Raphson, if $f'(x)$ is small near the root, how does this affect convergence?
 - (c) In numerical integration, how do rounding errors accumulate?
 - (d) Suggest strategies to minimize computational errors

Section H: Advanced Applications

37. Solve systems of equations numerically. For the system: $x^2 + y^2 = 5$, $xy = 2$
- (a) Rearrange to eliminate one variable
 - (b) Solve the resulting equation using Newton-Raphson
 - (c) Find all solutions
 - (d) Verify your answers by substitution
 - (e) Compare with algebraic solution
38. A projectile's height is given by $h(t) = 20t - 5t^2$ for $t \geq 0$.
- (a) Find when the projectile hits the ground exactly
 - (b) Use numerical methods to find when $h(t) = 10$
 - (c) Find the maximum height and when it occurs
 - (d) Use numerical integration to find the total distance traveled
 - (e) Model air resistance with $h(t) = 20t - 5t^2 - 0.1t^3$ and solve numerically
39. The equation $x^3 - 3x + a = 0$ has parameter a .
- (a) For what values of a does the equation have three real roots?
 - (b) For $a = 1$, find all roots numerically

- (c) For $a = -1$, find all roots numerically
 - (d) Investigate the behavior as a varies
 - (e) Create a bifurcation diagram showing how roots change with a
40. Population growth is modeled by $\frac{dP}{dt} = rP(1 - \frac{P}{K})$ where $P(0) = P_0$.
- (a) For $r = 0.1$, $K = 1000$, $P_0 = 50$, the solution is $P(t) = \frac{1000}{1+19e^{-0.1t}}$
 - (b) Use Newton-Raphson to find when $P(t) = 500$
 - (c) Find when the growth rate $\frac{dP}{dt}$ is maximum
 - (d) Use numerical integration to find the total growth in the first 20 years
 - (e) Model seasonal variation with $r(t) = 0.1 + 0.02 \sin(2\pi t)$ and solve numerically
41. Financial modeling: An investment grows according to $A(t) = Pe^{rt}$ where r varies.
- (a) If $P = 1000$ and $A(5) = 1500$, find r using Newton-Raphson
 - (b) For compound interest $A = P(1 + \frac{r}{n})^{nt}$, find r when $P = 1000$, $A = 2000$, $t = 10$, $n = 12$
 - (c) Use numerical integration to find the average value of $A(t)$ over $[0, 10]$
 - (d) Model variable interest rates and compare investment strategies

Section I: Advanced Topics and Optimization

42. Multi-variable Newton-Raphson for system: $f(x, y) = x^2 + y^2 - 4 = 0$, $g(x, y) = xy - 1 = 0$
- (a) Set up the Jacobian matrix
 - (b) Derive the iteration formulas
 - (c) Find a solution starting from $(1.5, 1.5)$
 - (d) Compare with single-variable approach
 - (e) Discuss convergence criteria for systems
43. Optimization using numerical methods:
- (a) Find the minimum of $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 2$ using Newton-Raphson on $f'(x) = 0$
 - (b) Use numerical integration to find the area under $f(x)$ from 0 to 3
 - (c) Find the point where $f(x) = 1$ has multiple solutions
 - (d) Analyze the stability of each critical point
44. Adaptive integration methods:
- (a) Implement Richardson extrapolation for trapezium rule
 - (b) Use adaptive Simpson's rule with error control
 - (c) Compare computational efficiency
 - (d) Apply to $\int_0^1 \frac{\sin x}{x} dx$ (using $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$)
45. Boundary value problems: Solve $y'' + y = x$ with $y(0) = 0$, $y(\pi) = 0$:
- (a) Convert to a system of first-order equations
 - (b) Use shooting method with Newton-Raphson
 - (c) Implement finite difference method
 - (d) Compare solutions with exact answer
 - (e) Discuss numerical stability

46. Fourier analysis using numerical methods:
- (a) For $f(x) = x^2$ on $[-\pi, \pi]$, compute Fourier coefficients numerically
 - (b) Use trapezium rule to evaluate $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$
 - (c) Compute b_n coefficients similarly
 - (d) Compare with analytical Fourier series
 - (e) Discuss convergence and Gibbs phenomenon
47. Chaos and sensitivity analysis:
- (a) Study the logistic map $x_{n+1} = rx_n(1 - x_n)$
 - (b) For $r = 3.2$, find the fixed point using Newton-Raphson
 - (c) For $r = 3.8$, demonstrate chaotic behavior
 - (d) Show sensitivity to initial conditions
 - (e) Create a bifurcation diagram for $r \in [2.5, 4]$
 - (f) Discuss implications for numerical accuracy
48. Monte Carlo methods:
- (a) Estimate π using random points in a unit square
 - (b) Use Monte Carlo integration for $\int_0^1 e^{-x^2} dx$
 - (c) Compare accuracy with deterministic methods
 - (d) Analyze convergence rate ($\propto 1/\sqrt{n}$)
 - (e) Discuss when Monte Carlo methods are preferred
 - (f) Apply to multi-dimensional integration
49. Numerical differentiation and applications:
- (a) Derive forward, backward, and central difference formulas
 - (b) Estimate $f'(1)$ for $f(x) = e^x$ using different step sizes
 - (c) Analyze truncation error vs. rounding error trade-off
 - (d) Apply to find critical points of tabulated data
 - (e) Use for solving differential equations numerically
 - (f) Implement higher-order difference formulas
50. Spline interpolation and curve fitting:
- (a) Construct cubic spline through points $(0, 1)$, $(1, 2)$, $(2, 0)$, $(3, 1)$
 - (b) Compare with polynomial interpolation
 - (c) Discuss advantages of splines for numerical integration
 - (d) Apply to data smoothing problems
 - (e) Use for solving differential equations
 - (f) Implement natural and clamped boundary conditions
51. Design a comprehensive numerical analysis project:
- (a) Choose a real-world problem requiring multiple numerical methods
 - (b) Implement root-finding, integration, and optimization
 - (c) Analyze error propagation and computational complexity
 - (d) Compare different numerical approaches
 - (e) Validate results against known solutions where possible
 - (f) Present findings with appropriate visualizations
 - (g) Discuss limitations and potential improvements

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 200

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