A Level Pure Mathematics
Practice Test 1: Differentiation

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Basic Differentiation - Polynomials

1. Differentiate these polynomial functions:
   (a) $f(x) = 3x^4 - 2x^3 + 5x^2 - 7x + 1$
   (b) $g(x) = 2x^5 + \frac{1}{2}x^3 - 4x + 6$
   (c) $h(x) = (x-1)(x+3)$
   (d) $k(x) = (2x-3)^2$
   (e) $p(x) = x^3(x^2 - 4)$
   (f) $q(x) = \frac{x^4 - 2x^2 + 1}{x}$
2. Find $\frac{dy}{dx}$ for:
   (a) $y = 3x^{-2} + 4x^{-1} - 2$
   (b) $y = \frac{2}{x^3} - \frac{5}{\sqrt{x}} + 3\sqrt{x}$
   (c) $y = 2\sqrt{x^3} + \frac{3}{x^2} - x^{-\frac{1}{2}}$
   (d) $y = (x + \frac{1}{x})^2$
3. Find the gradient of these curves at the given points:
   (a) $y = x^3 - 4x^2 + 6x - 2$ at $x = 2$
   (b) $y = 2x^4 - 3x^2 + 1$ at $x = -1$
   (c) $y = \frac{x^2+1}{x}$ at $x = 1$
   (d) $y = (x-2)^3$ at $x = 3$
4. Find the equation of the tangent line to:
   (a) $y = x^3 - 2x^2 + x - 1$ at the point where $x = 1$
   (b) $y = 2x^2 - 5x + 3$ at the point $(2, 1)$
   (c) $y = x^3 + x$ at the point where the gradient is 4
   (d) $y = \frac{x^2}{2} - 3x + 1$ at the point where $x = 4$
5. Given that $f(x) = ax^3 + bx^2 + cx + d$ and $f'(x) = 6x^2 - 12x + 3$:
   (a) Find the values of a, b, and c
   (b) If $f(0) = 5$, find the value of d
   (c) Write the complete expression for $f(x)$
   (d) Find $f(2)$ and $f'(2)$

Section B: Differentiation of Special Functions

6. Differentiate these exponential and logarithmic functions:
   (a) $f(x) = e^x$
   (b) $g(x) = 3e^x + 2x^2$
   (c) $h(x) = xe^x$
   (d) $k(x) = \ln x$
   (e) $p(x) = x \ln x$
   (f) $q(x) = \frac{\ln x}{x}$

7. Differentiate these trigonometric functions:
   (a) $f(x) = \sin x + \cos x$
   (b) $g(x) = 3 \sin x - 2 \cos x + x^2$
   (c) $h(x) = x \sin x$
   (d) $k(x) = \frac{\cos x}{x}$
   (e) $p(x) = \tan x$
   (f) $q(x) = \sec x$

8. Find $\frac{dy}{dx}$ for:
   (a) $y = e^{2x}$
   (b) $y = \ln(3x)$
   (c) $y = \sin(4x)$
   (d) $y = \cos(2x - 1)$
   (e) $y = e^{x^2}$
   (f) $y = \ln(x^2 + 1)$

9. Differentiate using appropriate rules:
   (a) $f(x) = e^x \cos x$
   (b) $g(x) = x^2 \sin x$
   (c) $h(x) = \frac{e^x}{x}$
   (d) $k(x) = \frac{\sin x}{\cos x}$
   (e) $p(x) = (\ln x)^2$
   (f) $q(x) = \sqrt{\sin x}$

10. Find the derivatives of:
   (a) $f(x) = \sin^2 x$
   (b) $g(x) = \cos^3 x$
   (c) $h(x) = e^{\sin x}$
   (d) $k(x) = \ln(\cos x)$
   (e) $p(x) = (\sin x + \cos x)^2$
   (f) $q(x) = \tan^{-1} x$ (inverse tan)

Section C: Product Rule and Quotient Rule

11. Use the product rule to differentiate:
    (a) $f(x) = (x^2 + 1)(x^3 - 2)$
    (b) $g(x) = (2x - 3)(x^2 + x + 1)$
    (c) $h(x) = x^2 e^x$
    (d) $k(x) = (x + 1)\ln x$
    (e) $p(x) = \sin x \cos x$
    (f) $q(x) = x^3 \sin x$

12. Use the quotient rule to differentiate:
    (a) $f(x) = \frac{x^2+1}{x-1}$
    (b) $g(x) = \frac{2x+3}{x^2+1}$
    (c) $h(x) = \frac{e^x}{x^2}$
    (d) $k(x) = \frac{\ln x}{x+1}$
    (e) $p(x) = \frac{\sin x}{1+\cos x}$
    (f) $q(x) = \frac{x^2}{\sin x}$

13. Choose the most appropriate method to differentiate:
    (a) $f(x) = \frac{x^3+2x}{x}$
    (b) $g(x) = (x^2 - 1)(x + 2)$
    (c) $h(x) = \frac{x^2+x+1}{x^2}$
    (d) $k(x) = x(x^2 + 1)^2$
    (e) $p(x) = \frac{(x+1)^2}{x}$
    (f) $q(x) = x^2\sqrt{x + 1}$

14. Given $f(x) = x^2$ and $g(x) = \sin x$:
    (a) Find $(fg)'(x)$ using the product rule
    (b) Find $(\frac{f}{g})'(x)$ using the quotient rule
    (c) Evaluate $(fg)'(\frac{\pi}{4})$
    (d) Evaluate $(\frac{f}{g})'(\frac{\pi}{6})$

15. Prove these differentiation rules:
    (a) Product rule: $(uv)' = u'v + uv'$
    (b) Quotient rule: $(\frac{u}{v})' = \frac{u'v-uv'}{v^2}$
    (c) Show that $(\frac{1}{v})' = -\frac{v'}{v^2}$
    (d) Verify that $(uvw)' = u'vw + uv'w + uvw'$

Section D: Chain Rule

16. Use the chain rule to differentiate:
    (a) $f(x) = (2x+1)^3$
    (b) $g(x) = (x^2 - 3x + 1)^4$
    (c) $h(x) = \sqrt{x^2+1}$
    (d) $k(x) = (3x-2)^{-2}$
    (e) $p(x) = \sin(2x+1)$
    (f) $q(x) = \cos(x^2)$
17. Find $\frac{dy}{dx}$ for:
    (a) $y = e^{3x-1}$
    (b) $y = \ln(2x+5)$
    (c) $y = (x^2+3x)^5$
    (d) $y = \sin^2 x$
    (e) $y = \cos(e^x)$
    (f) $y = e^{\sin x}$
18. Differentiate these composite functions:
    (a) $f(x) = (e^x+1)^3$
    (b) $g(x) = \ln(x^2+2x+1)$
    (c) $h(x) = \sin(\ln x)$
    (d) $k(x) = e^{x \cos x}$
    (e) $p(x) = (\sin x + \cos x)^4$
    (f) $q(x) = \ln(\sin x)$
19. Use multiple rules to differentiate:
    (a) $f(x) = x(2x+1)^3$
    (b) $g(x) = \frac{x^2}{(x+1)^2}$
    (c) $h(x) = x^2 \sin(3x)$
    (d) $k(x) = e^x \cos(2x)$
    (e) $p(x) = \frac{\ln x}{\sqrt{x}}$
    (f) $q(x) = \frac{(x^2+1)^3}{x}$
20. Find the second derivatives:
    (a) $f(x) = (x+1)^4$
    (b) $g(x) = \sin(2x)$
    (c) $h(x) = e^{-x}$
    (d) $k(x) = \ln(x^2)$
    (e) $p(x) = x^2 e^x$
    (f) $q(x) = \sin x \cos x$

Section E: Stationary Points

21. Find the coordinates of stationary points for:
    (a) $f(x) = x^3 - 3x^2 + 2$
    (b) $g(x) = 2x^3 - 9x^2 + 12x - 1$
    (c) $h(x) = x^4 - 4x^3 + 6x^2$
    (d) $k(x) = \frac{x^2}{x-1}$ for $x \neq 1$
22. Determine the nature of each stationary point using the second derivative test:
    (a) $f(x) = x^3 - 6x^2 + 9x + 1$
    (b) $g(x) = 2x^3 - 3x^2 - 12x + 5$
    (c) $h(x) = x^4 - 2x^2 + 3$
    (d) $k(x) = xe^{-x}$
23. Find and classify all stationary points:
    (a) $f(x) = x^3 - 3x + 2$
    (b) $g(x) = 2x^3 + 3x^2 - 12x + 1$
    (c) $h(x) = x^4 - 8x^2 + 16$
    (d) $k(x) = x + \frac{1}{x}$ for $x > 0$
24. For the function $f(x) = ax^3 + bx^2 + cx + d$:
    (a) Find the conditions on a, b, and c for the function to have two stationary points
    (b) If $f(x) = x^3 - 3x^2 + 3x + 1$, show it has no stationary points
    (c) Find the values of k for which $f(x) = x^3 - 3kx + 2$ has exactly one stationary point
25. Analyze the function $f(x) = \frac{x^2-4}{x}$:
    (a) Find the domain of $f(x)$
    (b) Find $f'(x)$ and locate stationary points
    (c) Determine the nature of stationary points
    (d) Find any asymptotes
    (e) Sketch the graph of $y = f(x)$

Section F: Rates of Change

26. A particle moves along a line with position $s(t) = t^3 - 6t^2 + 9t + 2$ meters at time t seconds.
    (a) Find the velocity $v(t)$ and acceleration $a(t)$
    (b) Find when the particle is at rest
    (c) Calculate the velocity and acceleration at $t = 2$
    (d) Determine when the acceleration is zero
    (e) Find the displacement between $t = 1$ and $t = 4$
27. The volume of a sphere is $V = \frac{4}{3}\pi r^3$. If the radius increases at a rate of 2 cm/s:
    (a) Find the rate of change of volume when $r = 5$ cm
    (b) Express $\frac{dV}{dt}$ in terms of r and $\frac{dr}{dt}$
    (c) When is the volume increasing at 100π cm^3/s?

   (d) Find the rate of change of surface area when $r = 3$ cm

28. A ladder 5 meters long leans against a vertical wall. The bottom slides away at 1.5 m/s.
    (a) Set up the relationship between distances
    (b) Find how fast the top slides down when the bottom is 3m from the wall
    (c) Find the rate of change of the angle with the ground
    (d) When is the top sliding down fastest?

29. Water flows into a conical tank (vertex down) at 2 m^3/min. The tank has height 6m and radius 3m.
    (a) Express the volume in terms of height h
    (b) Find how fast the water level rises when $h = 2$m
    (c) Find the rate of change of radius when $h = 4$m
    (d) When is the water level rising fastest?

30. The population of a town grows according to $P(t) = 10000e^{0.02t}$ where t is years.
    (a) Find the growth rate $\frac{dP}{dt}$
    (b) Calculate the population and growth rate after 5 years
    (c) When is the population growing at 250 people per year?
    (d) Express the growth rate as a percentage of current population

Section G: Optimization Problems

31. A farmer has 240m of fencing to enclose a rectangular field against a straight river (no fence needed along river).
    (a) Express the area in terms of one variable
    (b) Find the dimensions for maximum area
    (c) Calculate the maximum area
    (d) Verify this is a maximum using the second derivative

32. A rectangular box with square base has volume 32 m^3. The material for the base costs £5/m^2, sides cost £3/m^2, and top costs £2/m^2.
    (a) Express the cost in terms of the base side length
    (b) Find dimensions for minimum cost
    (c) Calculate the minimum cost
    (d) Find the ratio of height to base side length

33. A company's profit function is $P(x) = -2x^3 + 30x^2 + 72x - 100$ thousand pounds, where x is production level (thousands of units).
    (a) Find the production levels for maximum and minimum profit
    (b) Calculate the maximum profit
    (c) Find the marginal profit function
    (d) Determine the optimal production level

34. A window consists of a rectangle topped by a semicircle. The perimeter is 12m.
    (a) Express the area in terms of the rectangle width

(b) Find dimensions for maximum area
(c) Calculate the maximum area
(d) Find the ratio of rectangle height to width

35. A cylindrical can must hold 500 cm³. Find dimensions to minimize surface area.

(a) Express surface area in terms of radius
(b) Find the critical points
(c) Determine optimal radius and height
(d) Calculate minimum surface area
(e) Verify this gives a minimum

Section H: Implicit Differentiation and Related Rates

36. Find $\frac{dy}{dx}$ using implicit differentiation:

(a) $x^2 + y^2 = 25$
(b) $x^2 + 2xy + y^2 = 16$
(c) $x^3 + y^3 = 6xy$
(d) $\sin(xy) = x + y$
(e) $e^{xy} = x + y$
(f) $\ln(xy) = x - y$

37. Find the equation of the tangent to these curves at the given points:

(a) $x^2 + y^2 = 13$ at $(2, 3)$
(b) $x^2 - xy + y^2 = 7$ at $(1, 2)$
(c) $x^3 + y^3 = 2$ at $(1, 1)$
(d) $xe^y = 2$ at $(2, 0)$

38. Use implicit differentiation to find $\frac{d^2y}{dx^2}$:

(a) $x^2 + y^2 = 1$
(b) $xy = 1$
(c) $x^2 - y^2 = 4$

39. Two cars start from the same point. Car A travels north at 60 km/h, Car B travels east at 80 km/h.

(a) Express the distance between cars as a function of time
(b) Find how fast they're separating after 2 hours
(c) When are they separating at 120 km/h?
(d) Find the minimum distance between them

40. A balloon is inflated so its volume increases at 100 cm³/s. Find the rate of increase of:

(a) Radius when $r = 5$ cm
(b) Surface area when $r = 10$ cm
(c) Diameter when volume is 1000 cm³
(d) The rate when surface area is 400π cm²

Section I: Advanced Applications

41. A Norman window has the shape of a rectangle topped by a semicircle, with total perimeter 20m.

    (a) Find dimensions to maximize the area
    (b) Calculate the maximum area
    (c) Find the optimal ratio of rectangle height to width
    (d) Determine what fraction of area is rectangular

42. The strength of a rectangular beam is proportional to wd^2 where w is width and d is depth. A beam is cut from a circular log of radius 12 cm.

    (a) Express strength in terms of width w
    (b) Find dimensions for maximum strength
    (c) Calculate the ratio $\frac{d}{w}$ for strongest beam
    (d) Compare with beam of square cross-section

43. A drug concentration in blood follows $C(t) = \frac{At}{(t+1)^2}$ mg/L where t is hours after injection.

    (a) Find when concentration is maximum
    (b) If peak concentration is 4 mg/L, find A
    (c) Calculate the rate of change at $t = 1$
    (d) Find when concentration is decreasing fastest
    (e) Determine the half-life from peak concentration

44. A sector of a circle with radius r and central angle θ (in radians) has perimeter 20 cm.

    (a) Express the area in terms of r
    (b) Find r and θ for maximum area
    (c) Calculate the maximum area
    (d) Show the optimal angle is 2 radians

45. A factory's total cost is $C(x) = 0.1x^3 - 1.2x^2 + 6x + 100$ for producing x hundred units.

    (a) Find the marginal cost function
    (b) Determine the production level for minimum average cost
    (c) Calculate the minimum average cost
    (d) Find when marginal cost equals average cost
    (e) Graph the cost functions and interpret economically

46. Two positive numbers have sum 20. Find the numbers that:

    (a) Maximize their product
    (b) Minimize the sum of their squares
    (c) Maximize the sum of their square roots
    (d) Minimize $x^2 + y^3$ where $x + y = 20$
    (e) Explain why the answers differ

47. A projectile follows the path $y = x\tan\alpha - \frac{gx^2}{2v^2\cos^2\alpha}$ where α is launch angle.

    (a) Find the range (horizontal distance when $y = 0$)

(b) Find the angle for maximum range
(c) Calculate the maximum height
(d) Find the angle for maximum height at distance x
(e) Derive the envelope of trajectories

48. Design a calculus-based model for a real-world optimization problem:

(a) Define your scenario and variables clearly
(b) Set up the objective function and constraints
(c) Use differentiation to find optimal solutions
(d) Verify your solution makes physical sense
(e) Discuss limitations of your model

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

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