A Level Pure Mathematics
Practice Test 2: Differential Equations

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Fundamentals and Classification

1. Explain the following concepts with examples:
   (a) Autonomous vs. non-autonomous differential equations
   (b) Boundary conditions vs. initial conditions
   (c) Explicit vs. implicit solutions
   (d) Direction fields and isoclines
   (e) Singular solutions
   (f) Equilibrium points and stability

2. Determine the order, degree, and linearity of these differential equations:
   (a) $\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} = y^2$
   (b) $\left(\frac{d^3y}{dx^3}\right)^2 + y = x^2$
   (c) $\sin\left(\frac{dy}{dx}\right) + xy = 0$
   (d) $\frac{dy}{dx} + P(x)y = Q(x)y^n$ (Bernoulli equation)
   (e) $x^2\frac{d^2y}{dx^2} + xy\frac{dy}{dx} + y = 0$
   (f) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

3. Verify by substitution that these are solutions:
   (a) $y = Ce^{x^2}$ satisfies $\frac{dy}{dx} = 2xy$
   (b) $y = x^2 + \frac{C}{x}$ satisfies $x\frac{dy}{dx} + y = 3x^2$
   (c) $y = A\sin(t) + B\cos(t)$ satisfies $\frac{d^2y}{dt^2} +^2 y = 0$
   (d) $y = e^{-x}(C_1 + C_2x)$ satisfies $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

4. Form differential equations from these families of curves:
   (a) $y = Ce^{-2x}$ (one parameter)
   (b) $y = C_1e^x + C_2e^{-x}$ (two parameters)
   (c) $x^2 + y^2 = C$ (circles centered at origin)

(d) $y = Cx^3$ (cubic curves through origin)

5. Analyze the behavior of solutions:
   (a) Sketch direction fields for $\frac{dy}{dx} = x - y$
   (b) Identify equilibrium solutions for $\frac{dy}{dx} = y(2 - y)$
   (c) Determine stability of equilibria in $\frac{dy}{dx} = y^2 - 4$
   (d) Discuss long-term behavior of solutions to $\frac{dy}{dx} = -2y$

Section B: Direct Integration Methods

6. Solve by direct integration:
   (a) $\frac{dy}{dx} = 4x^3 - 3x + 2$
   (b) $\frac{dy}{dx} = e^{-3x}$
   (c) $\frac{dy}{dx} = \frac{2}{3x+1}$
   (d) $\frac{dy}{dx} = \sin(4x) - \cos x$
   (e) $\frac{dy}{dx} = \frac{2x}{x^2+9}$
   (f) $\frac{dy}{dx} = xe^{x^2/2}$

7. Find particular solutions with given conditions:
   (a) $\frac{dy}{dx} = 9x^2 - 4x$, $y(1) = 3$
   (b) $\frac{dy}{dx} = 3e^{2x}$, $y(0) = 2$
   (c) $\frac{dy}{dx} = \cos(3x)$, $y(\pi/6) = 1$
   (d) $\frac{dy}{dx} = \frac{3}{x+2}$, $y(0) = \ln 2$ (for $x > -2$)
   (e) $\frac{dy}{dx} = x\sqrt{x^2 + 4}$, $y(0) = 5$

8. Higher-order problems:
   (a) $\frac{d^2y}{dx^2} = 8x - 4$, $y(0) = 1$, $y'(0) = 3$
   (b) $\frac{d^2y}{dx^2} = 2e^x$, $y(0) = 0$, $y'(0) = 1$
   (c) $\frac{d^3y}{dx^3} = 12$, $y(0) = 0$, $y'(0) = 2$, $y''(0) = -1$
   (d) $\frac{d^2y}{dx^2} = \cos x$, $y(0) = 1$, $y(\pi) = 0$

9. Applications:
   (a) A particle's acceleration is $a = 6t - 4$. Find velocity and position if $v(0) = 1$ and $s(0) = 2$.
   (b) An object falls from rest at height 80m. Find position and velocity after 4 seconds.
   (c) The gradient of a curve at any point equals $4x^3 - 2x$. Find the curve through $(1, 2)$.
   (d) A spring satisfies $\frac{d^2x}{dt^2} = -9x$. Find $x(t)$ if $x(0) = 1$ and $\dot{x}(0) = 3$.

10. Rate problems:
    (a) Water flows into a tank at rate $\frac{dV}{dt} = 5t + 2$ L/min. Find volume after 10 minutes if initially empty.
    (b) A population grows at rate $\frac{dP}{dt} = 100e^{0.02t}$. Find population growth in first 5 years.
    (c) Temperature increases at rate $\frac{dT}{dt} = 3\cos(t/2)$ degrees/hour. Find temperature change over 4 hours.
    (d) Investment grows at rate $\frac{dA}{dt} = 0.06A + 1000$. Solve for A(t).

Section C: Separation of Variables

11. Solve these separable equations:
    (a) $\frac{dy}{dx} = 2xy^2$
    (b) $\frac{dy}{dx} = \frac{3x}{y}$
    (c) $\frac{dy}{dx} = e^{x-2y}$
    (d) $\frac{dy}{dx} = \frac{y^2}{x^2+4}$
    (e) $\frac{dy}{dx} = \frac{\cos x}{\sin y}$
    (f) $\frac{dy}{dx} = \frac{x^2 y}{x^3+1}$
12. Find particular solutions:
    (a) $\frac{dy}{dx} = 3xy$, $y(0) = 2$
    (b) $\frac{dy}{dx} = \frac{2y}{x}$, $y(1) = 3$ (for $x > 0$)
    (c) $\frac{dy}{dx} = \frac{x^3}{y^2}$, $y(0) = 2$
    (d) $\frac{dy}{dx} = y(3-y)$, $y(0) = 1$
    (e) $\frac{dy}{dx} = \frac{x}{\sqrt{4-y^2}}$, $y(0) = 0$
13. Advanced separable equations:
    (a) $(1+y^2)\frac{dy}{dx} = 2xy$
    (b) $\frac{dy}{dx} = \frac{ye^{2x}}{x^2+1}$
    (c) $\cos^2 y \frac{dy}{dx} = \sin x$
    (d) $\frac{dy}{dx} = \frac{x^3(1+y^2)}{y(1+x^4)}$
    (e) $y \ln y \frac{dy}{dx} = 2x$
14. Applications:
    (a) Population growth: $\frac{dP}{dt} = 0.03P$, $P(0) = 500$. Find $P(t)$ and doubling time.
    (b) Radioactive decay: $\frac{dN}{dt} = -0.05N$. If $N(0) = 200g$, find amount after 20 years.
    (c) Newton's cooling: $\frac{dT}{dt} = -k(T-25)$. Object cools from 90°C to 70°C in 3 minutes.
    (d) Chemical reaction: $\frac{dx}{dt} = k(10-x)^2$ with $x(0) = 0$.
15. Test for separability:
    (a) $\frac{dy}{dx} = x^2 + y^2$ (not separable)
    (b) $\frac{dy}{dx} = x^2 y + 2xy$ (separable)
    (c) $\frac{dy}{dx} = \frac{2x+3y}{x-y}$ (not separable)
    (d) $\frac{dy}{dx} = e^{2x+3y}$ (separable)
    (e) $\frac{dy}{dx} = xy + x$ (separable)

Section D: Linear First-Order Equations

16. Solve using integrating factors:
    (a) $\frac{dy}{dx} + 4y = e^{3x}$
    (b) $\frac{dy}{dx} - 2y = x^2$
    (c) $\frac{dy}{dx} + \frac{3y}{x} = x^3$ (for $x > 0$)
    (d) $\frac{dy}{dx} + y \sin x = \cos x \sin x$
    (e) $x\frac{dy}{dx} + 2y = x^3$
    (f) $\frac{dy}{dx} + 3xy = 2xe^{-3x^2/2}$
17. Solve with initial conditions:
    (a) $\frac{dy}{dx} + 3y = 9e^x$, $y(0) = 1$
    (b) $\frac{dy}{dx} - y = 2x$, $y(0) = 4$
    (c) $\frac{dy}{dx} + 2y = 6$, $y(0) = 0$
    (d) $\frac{dy}{dx} + \frac{y}{x} = 2x$, $y(1) = 3$ (for $x > 0$)
18. More complex linear equations:
    (a) $\frac{dy}{dx} + y \cot x = \csc x$
    (b) $(1 + x^2)\frac{dy}{dx} + 2xy = 2(1 + x^2)$
    (c) $\frac{dy}{dx} + \frac{2y}{x^2+1} = \frac{2x}{x^2+1}$
    (d) $x^2\frac{dy}{dx} + 3xy = 2x^4$ (for $x > 0$)
19. Applications:
    (a) RL circuit: $L\frac{di}{dt} + Ri = V_0$ with constant voltage. Find current $i(t)$.
    (b) Mixing: Pure water flows into a 100L tank containing salt solution at 2 L/min. Find salt concentration.
    (c) Investment: $\frac{dA}{dt} = 0.04A - 800$ (4% interest, £800 withdrawal). Solve for A(t).
    (d) Falling object: $m\frac{dv}{dt} + kv = mg$ with air resistance.
20. Method comparison:
    (a) Solve $\frac{dy}{dx} = 2xy + 2x$ by separation
    (b) Solve as linear equation: $\frac{dy}{dx} - 2xy = 2x$
    (c) Verify solutions are equivalent
    (d) Discuss when each method is preferable

Section E: Second-Order Linear Equations - Homogeneous

21. Find auxiliary equations and solve:
    (a) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$
    (b) $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0$
    (c) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0$
    (d) $\frac{d^2y}{dx^2} + 36y = 0$
    (e) $\frac{d^2y}{dx^2} - 16y = 0$

(f) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 10y = 0$

22. Classify roots and write general solutions:

(a) $m^2 - 5m + 6 = 0$ (distinct real roots)
(b) $m^2 + 4m + 4 = 0$ (repeated real root)
(c) $m^2 + 2m + 10 = 0$ (complex conjugate roots)
(d) $m^2 - 25 = 0$ (distinct real roots)
(e) $m^2 + 4 = 0$ (pure imaginary roots)

23. Solve with initial conditions:

(a) $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$, $y(0) = 1$, $y'(0) = 2$
(b) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$, $y(0) = 2$, $y'(0) = -3$
(c) $\frac{d^2y}{dx^2} + 16y = 0$, $y(0) = 0$, $y'(0) = 4$
(d) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$, $y(0) = 1$, $y'(0) = 1$

24. Analyze solution behavior:

(a) For $\frac{d^2y}{dx^2} - k^2y = 0$, describe exponential solutions
(b) For $\frac{d^2y}{dx^2} +^2 y = 0$, explain oscillatory behavior
(c) Compare overdamped, critically damped, and underdamped motion
(d) Sketch typical solution curves for each case

25. Higher-order equations:

(a) $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$
(b) $\frac{d^4y}{dx^4} - 81y = 0$
(c) Discuss solution structure for nth order equations

Section F: Second-Order Linear Equations - Non-homogeneous

26. Method of undetermined coefficients:

(a) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 12$
(b) $\frac{d^2y}{dx^2} + 16y = 32x$
(c) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = e^{3x}$
(d) $\frac{d^2y}{dx^2} + 9y = \cos(2x)$
(e) $\frac{d^2y}{dx^2} - 9y = 3e^{-3x}$
(f) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = x^2 - 2$

27. Handle resonance cases:

(a) $\frac{d^2y}{dx^2} + 9y = \sin(3x)$ (resonance)
(b) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{2x}$ (resonance)
(c) $\frac{d^2y}{dx^2} + 4y = \cos(2x)$ (resonance)
(d) Explain modification needed for resonance

28. Complete solutions with initial conditions:

(a) $\frac{d^2y}{dx^2} + 4y = 8$, $y(0) = 1$, $y'(0) = 0$
(b) $\frac{d^2y}{dx^2} - 4y = 4x$, $y(0) = 0$, $y'(0) = 2$
(c) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 2e^{-x}$, $y(0) = 1$, $y'(0) = 0$

29. Trial solutions guide:
    (a) For $f(x) =$ constant, polynomial, exponential, trigonometric
    (b) When to multiply by x (resonance cases)
    (c) Products like xe^{ax}, $x\sin(ax)$
    (d) Combinations of different function types

30. Variation of parameters:
    (a) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x}$
    (b) $\frac{d^2y}{dx^2} + 4y = \tan(2x)$
    (c) Compare with undetermined coefficients method

Section G: Applications of Second-Order Equations

31. Mechanical vibrations:
    (a) Simple harmonic motion: $m\frac{d^2x}{dt^2} + kx = 0$ with $x(0) = 3$, $\dot{x}(0) = 0$
    (b) Find period, frequency, and amplitude for $m = 2$ kg, $k = 18$ N/m
    (c) Maximum kinetic and potential energy
    (d) Phase relationship between displacement and velocity

32. Damped oscillations:
    (a) $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$ with $m = 1$, $c = 5$, $k = 6$ (overdamped)
    (b) Critical damping: $m = 1$, $c = 6$, $k = 9$ with $x(0) = 2$, $\dot{x}(0) = -3$
    (c) Underdamped: $m = 1$, $c = 3$, $k = 10$ with $x(0) = 1$, $\dot{x}(0) = 0$
    (d) Calculate damping ratio and logarithmic decrement

33. Forced vibrations:
    (a) $\frac{d^2x}{dt^2} + 25x = 50\cos(4t)$ with zero initial conditions
    (b) Find steady-state amplitude and phase lag
    (c) Resonance: $\frac{d^2x}{dt^2} + 16x = 32\cos(4t)$
    (d) Beating phenomenon when forcing frequency is near natural frequency

34. Electrical circuits:
    (a) RLC series circuit: $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V(t)$
    (b) With $L = 1$ H, $R = 4$, $C = 0.25$ F, $V = 20$ V constant
    (c) Find charge $q(t)$ and current $i(t) = \frac{dq}{dt}$
    (d) Natural frequency and Q-factor analysis

35. Population and economic models:
    (a) Population with memory effects: $\frac{d^2P}{dt^2} + a\frac{dP}{dt} + bP = cK$
    (b) Business cycle model: $\frac{d^2Y}{dt^2} + \frac{dY}{dt} + Y = G$ (national income)
    (c) Stability analysis of equilibrium solutions
    (d) Phase plane interpretation

Section H: Advanced Methods and Special Cases

36. Homogeneous first-order equations:
    (a) $\frac{dy}{dx} = \frac{2x+y}{x}$ (substitute $v = \frac{y}{x}$)
    (b) $\frac{dy}{dx} = \frac{x^2-y^2}{2xy}$
    (c) $(x^2 - y^2)dx + 2xydy = 0$
    (d) Test: $\frac{dy}{dx} = \frac{ax+by}{cx+dy}$ for homogeneity
37. Bernoulli equations:
    (a) $\frac{dy}{dx} + 3y = xy^3$ (substitute $v = y^{1-n}$)
    (b) $x\frac{dy}{dx} + 2y = y^2$
    (c) $\frac{dy}{dx} - \frac{2y}{x} = \frac{y^3}{x^2}$
38. Exact differential equations:
    (a) $(3x^2 + 2y)dx + (2x + 4y)dy = 0$
    (b) $(e^x + y)dx + (x + e^y)dy = 0$
    (c) Find integrating factors when not exact
39. Reduction of order:
    (a) $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = 0$ (substitute $v = \frac{dy}{dx}$)
    (b) $y\frac{d^2y}{dx^2} = 2\left(\frac{dy}{dx}\right)^2$
    (c) Euler equations: $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$
40. Systems of equations:
    (a) $\frac{dx}{dt} = 3x + y$, $\frac{dy}{dt} = x + 3y$
    (b) Convert second-order to first-order system
    (c) Eigenvalue analysis for stability
    (d) Phase plane portraits

Section I: Modeling and Real-World Applications

41. Choose and solve one comprehensive modeling project:
    (a) Epidemic modeling: SIR model with vaccination
    (b) Predator-prey dynamics: Lotka-Volterra equations
    (c) Chemical kinetics: consecutive reactions
    (d) Climate modeling: temperature variations
    (e) Financial mathematics: interest rate models
    (f) Engineering control systems

    For your chosen project:
    (a) Derive the differential equation from first principles
    (b) Classify and solve using appropriate methods
    (c) Interpret solutions in physical context
    (d) Validate against real data where possible

(e) Discuss model limitations and improvements

42. Numerical methods:
    (a) Apply Euler's method to $\frac{dy}{dx} = y - x$, $y(0) = 1$
    (b) Compare with analytical solution
    (c) Discuss accuracy and stability
    (d) When are numerical methods necessary?

43. Boundary value problems:
    (a) $\frac{d^2y}{dx^2} + y = 0$ with $y(0) = y() = 0$
    (b) Find eigenvalues and eigenfunctions
    (c) Physical interpretation (vibrating string, heat conduction)

44. Existence and uniqueness:
    (a) State conditions for existence and uniqueness
    (b) Examples where solutions don't exist or aren't unique
    (c) Picard's theorem and successive approximations

45. Summary and review:
    (a) Classification flowchart for differential equations
    (b) Summary of solution methods
    (c) Common pitfalls and how to avoid them
    (d) Connections to other areas of mathematics

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 250

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