A Level Pure Mathematics
Practice Test 4: Exponentials and Logarithms

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Laws of Indices and Exponential Functions

1. Simplify these expressions using laws of indices:
   (a) $6^3 \times 6^4 \times 6^{-5}$
   (b) $\frac{4^9 \times 4^{-3}}{4^5}$
   (c) $(7^3)^2 \times 7^{-4}$
   (d) $\frac{(6^3)^2 \times 6^{-4}}{6^3}$
   (e) $(5^2 \times 5^{-3})^3$
   (f) $\frac{8^{3x} \times 8^{x+1}}{8^{2x-3}}$
2. Solve these exponential equations:
   (a) $7^x = 343$
   (b) $6^{x-3} = 216$
   (c) $3^{2x+1} = 243$
   (d) $16^x = \frac{1}{2}$
   (e) $36^x = 6^{x+4}$
   (f) $9^{2x} = 27^{x-1}$
3. Express in the form a^x where a is a rational number:
   (a) $(\frac{1}{4})^x \times 16^x$
   (b) $\frac{64^x}{4^{3x}}$
   (c) $(27)^{\frac{x}{3}} \times (\frac{1}{3})^x$
   (d) $\frac{64^x \times 8^{-2x}}{512^{\frac{x}{3}}}$
4. Sketch the graphs of these exponential functions, showing key features:
   (a) $y = 6^x$
   (b) $y = (\frac{1}{6})^x$
   (c) $y = 3^x - 2$
   (d) $y = 4^{x+3}$

(e) $y = -4^x$

(f) $y = 4^{-x}$

5. For the function $f(x) = 4e^x$:

(a) State the domain and range

(b) Find the y-intercept

(c) Describe the behavior as $x \to \infty$ and $x \to -\infty$

(d) Find $f'(x)$ and comment on the gradient

(e) Sketch the graph, showing the tangent at $(0, 4)$

Section B: Logarithmic Functions and Properties

6. Express these in logarithmic form:

(a) $7^2 = 49$

(b) $10^{-4} = 0.0001$

(c) $e^w = 3$

(d) $8^0 = 1$

(e) $9^{-1} = \frac{1}{9}$

(f) $d^u = t$

7. Express these in exponential form:

(a) $\log_7 343 = 3$

(b) $\log_{10} 0.0001 = -4$

(c) $\ln e^4 = 4$

(d) $\log_2 \frac{1}{32} = -5$

(e) $\log_d t = u$

(f) $\ln e^6 = 6$

8. Evaluate these logarithms without a calculator:

(a) $\log_7 49$

(b) $\log_6 216$

(c) $\log_{10} 1000000$

(d) $\log_5 \frac{1}{625}$

(e) $\log_{32} 8$

(f) $\log_{36} 6$

9. Use the laws of logarithms to simplify:

(a) $\log_d 6 + \log_d 8$

(b) $\log_d 42 - \log_d 6$

(c) $6 \log_d 3$

(d) $\log_d a + 5 \log_d b - \log_d c$

(e) $\frac{1}{5} \log_d 32 + \log_d 4$

(f) $\log_d(w^2 - 36) - \log_d(w - 6)$ where $w > 6$

10. Sketch the graphs of these logarithmic functions:

(a) $y = \log_7 x$
(b) $y = \ln x$
(c) $y = \log_7 x - 3$
(d) $y = \log_7(x + 3)$
(e) $y = -\log_7 x$
(f) $y = \log_7(-x)$ for $x < 0$

Section C: Solving Logarithmic Equations

11. Solve these logarithmic equations:

(a) $\log_7 x = 3$
(b) $\log_6(x - 3) = 2$
(c) $\log_{10}(5x + 1) = 2$
(d) $\ln(x + 5) = 0$
(e) $\log_2(x^2) = 6$
(f) $5 \log_3 x = 10$

12. Solve these equations involving multiple logarithms:

(a) $\log_7 x + \log_7 6 = 2$
(b) $\log_d 18 - \log_d 6 = \log_d x$
(c) $\log_3 x + \log_3(x - 8) = 2$
(d) $\log_{10} x - \log_{10}(x - 6) = \log_{10} 5$
(e) $5 \log_2 x = \log_2 8$
(f) $\log_5(x + 4) + \log_5(x - 4) = 2$

13. Solve these equations where the base is unknown:

(a) $\log_a 49 = 2$
(b) $\log_a \frac{1}{125} = -3$
(c) $\log_a 256 = \frac{8}{3}$
(d) $\log_a 729 = \frac{6}{2}$

14. Solve these quadratic logarithmic equations:

(a) $(\log_7 x)^2 = 9$
(b) $(\log_4 x)^2 - 6 \log_4 x + 8 = 0$
(c) $\log^2 x - 6 \log x + 8 = 0$ (base 10)
(d) $(\ln x)^2 - 8 \ln x + 15 = 0$

15. Use the change of base formula to evaluate:

(a) $\log_6 11$ in terms of natural logarithms
(b) $\log_8 24$ in terms of common logarithms
(c) $\log_5 18$ using ln
(d) Express $\log_r s \times \log_s t \times \log_t r$

Section D: Combined Exponential and Logarithmic Equations

16. Solve these mixed equations:
    (a) $e^x = 11$
    (b) $7^x = 35$
    (c) $8 \times 3^x = 72$
    (d) $4^{x+1} = 32$
    (e) $e^{2x} - 6e^x + 8 = 0$
    (f) $7^{2x} - 8 \times 7^x + 7 = 0$

17. Solve using substitution methods:
    (a) $36^x - 6^{x+3} - 216 = 0$ (let $y = 6^x$)
    (b) $32^x - 7 \times 2^x + 2 = 0$ (let $u = 2^x$)
    (c) $e^{2x} - 9e^x + 20 = 0$ (let $t = e^x$)
    (d) $\log^2 x - 6 \log x + 8 = 0$ (let $z = \log x$)

18. Find the exact solutions:
    (a) $\ln x + \ln(x + 5) = \ln 14$
    (b) $\log_3 x + \log_2 7x = 3$
    (c) $e^x + e^{-x} = 6$
    (d) $5 \ln x = \ln(x + 20)$

19. Solve these equations involving both exponentials and logarithms:
    (a) $x = \log_7(7^x + 6)$
    (b) $e^{\ln x} = x + 7$
    (c) $\ln(e^x - 4) = 4$
    (d) $\log_5(5^x + 20) = x + 2$

20. Find the values of x for which:
    (a) $7^x > 2401$
    (b) $\log_6 x < 2$
    (c) $e^x \leq 25$
    (d) $\ln x \geq 3$
    (e) $\log_7(x - 4) > 2$
    (f) $6^{x-1} < \frac{1}{36}$

Section E: Exponential Growth and Decay

21. A population grows according to $P = P_0 e^{kt}$ where $P_0 = 1600$ and $k = 0.06$ per year.
    (a) Find the population after 4 years
    (b) How long for the population to increase by 50%?
    (c) What is the percentage growth rate per year?
    (d) Find when the population reaches 5000
    (e) Calculate the population after 18 years

22. A radioactive substance decays according to $A = A_0 e^{-\lambda t}$ where $\lambda = 0.0578$ per year.
    (a) If initially there are 200g, find the amount after 8 years
    (b) Calculate the half-life of the substance
    (c) How long for 75% to decay?
    (d) What percentage remains after 25 years?
    (e) Find when only 25g remains

23. An investment grows at 9% compound interest per annum.
    (a) Write the growth formula
    (b) How long to increase by 150%?
    (c) If £2500 is invested, find the value after 14 years
    (d) How long for the investment to reach £12500?
    (e) Compare with simple interest at 9% over 8 years

24. The temperature of a cooling object follows Newton's law: $T = T_{\text{room}} + (T_0 - T_{\text{room}})e^{-kt}$
    (a) If room temperature is 22°C, initial temperature is 92°C, and $k = 0.08$ per minute, find the temperature after 15 minutes
    (b) How long for the object to cool to 45°C?
    (c) Find the temperature after 50 minutes
    (d) What happens as $t \to \infty$?
    (e) If the object cools to 65°C after 8 minutes, find k

25. Carbon-14 dating uses the formula $A = A_0 e^{-0.000121t}$ where t is in years.
    (a) Calculate the half-life of carbon-14
    (b) If a sample has 40% of its original carbon-14, find its age
    (c) How old is a sample with 5% remaining?
    (d) What percentage remains after 25000 years?
    (e) Find the age of a sample with ratio 0.55 of living organisms

Section F: Logarithmic Modeling and Applications

26. The Richter scale for earthquake magnitude is given by $M = \log_{10}(\frac{I}{I_0})$ where I is intensity.
    (a) If one earthquake has magnitude 6.5 and another has magnitude 4.5, compare their intensities
    (b) An earthquake has intensity 100000 times the reference level. Find its magnitude
    (c) How much more intense is a magnitude 8.5 earthquake than magnitude 6.5?
    (d) Find the magnitude of an earthquake with intensity $2 \times 10^7 I_0$

27. The pH scale is defined as $\text{pH} = -\log_{10}[\text{H}^+]$ where $[\text{H}^+]$ is hydrogen ion concentration.
    (a) Find the pH when $[\text{H}^+] = 10^{-6}$ mol/L
    (b) If pH = 2.5, find the hydrogen ion concentration
    (c) Compare the acidity of solutions with pH 1 and pH 8
    (d) Find the pH when $[\text{H}^+] = 6.3 \times 10^{-2}$ mol/L
    (e) If the concentration is divided by 4, how does the pH change?

28. Sound intensity level in decibels is $L = 10\log_{10}(\frac{I}{I_0})$ where $I_0 = 10^{-12}$ W/m².
    (a) Find the decibel level when $I = 10^{-3}$ W/m²
    (b) A sound has level 95 dB. Find its intensity
    (c) How much more intense is 110 dB than 70 dB?
    (d) Find the intensity of a 35 dB sound
    (e) If intensity increases by factor 2000, by how much do decibels increase?

29. The Michaelis-Menten equation in biochemistry is $v = \frac{V_{\max}[S]}{K_m+[S]}$.
    (a) Take logarithms to linearize when $[S] >> K_m$
    (b) If $V_{\max} = 150$, $K_m = 12$, find v when $[S] = 24$
    (c) Plot $\log v$ against $\log[S]$ for large $[S]$
    (d) Find $[S]$ when $v = \frac{2V_{\max}}{3}$

30. In information theory, entropy is $H = -\sum p_i \log_2 p_i$.
    (a) For a fair 16-sided die, calculate the entropy
    (b) For a biased coin with $P(\text{H}) = 0.9$, find the entropy
    (c) Find the entropy of a fair 32-sided die
    (d) What probability distribution maximizes entropy for 6 outcomes?

Section G: Advanced Functions and Transformations

31. Analyze the function $f(x) = \ln(x+5) - 4$:
    (a) State the domain and range
    (b) Find the x and y intercepts
    (c) Identify any asymptotes
    (d) Find $f^{-1}(x)$
    (e) Sketch both $f(x)$ and $f^{-1}(x)$

32. For the function $g(x) = e^{5x-3} - 6$:
    (a) Describe the transformations from $y = e^x$
    (b) State the domain and range
    (c) Find the horizontal asymptote
    (d) Solve $g(x) = 0$
    (e) Find $g^{-1}(x)$

33. Consider $h(x) = \log_5(25 - x^2)$:
    (a) Find the domain of $h(x)$
    (b) Determine the range
    (c) Find the maximum value and where it occurs
    (d) Solve $h(x) = 1$
    (e) Sketch the graph of $y = h(x)$

34. The function $k(x) = re^{sx} + v$ passes through $(0, 11)$, $(1, 18)$, and has horizontal asymptote $y = 5$.
    (a) Find the values of r, s, and v

(b) Write the equation of $k(x)$
(c) Find $k(2)$
(d) Solve $k(x) = 25$
(e) Find the domain and range of $k(x)$

35. Investigate the function $m(x) = x^3 \ln x$ for $x > 0$:
    (a) Find $m'(x)$ and $m''(x)$
    (b) Locate any stationary points
    (c) Determine the nature of stationary points
    (d) Find the behavior as $x \to 0^+$ and $x \to \infty$
    (e) Sketch the graph of $y = m(x)$

Section H: Systems and Complex Problems

36. Solve these simultaneous equations:
    (a) $\begin{cases} y = 5^x \\ y = 10 - x \end{cases}$
    (b) $\begin{cases} \ln y = 5x \\ y = e^{x+4} \end{cases}$
    (c) $\begin{cases} \log_5 x + \log_5 y = 3 \\ x + y = 20 \end{cases}$
    (d) $\begin{cases} e^x + e^y = 12 \\ e^x - e^y = 8 \end{cases}$

37. Find where these curves intersect:
    (a) $y = e^x$ and $y = \ln x$
    (b) $y = 5^x$ and $y = x^5$
    (c) $y = \log x$ and $y = 5 - x$
    (d) $y = e^{-x}$ and $y = x + 4$

38. Solve these differential equations:
    (a) $\frac{dy}{dx} = ky$ where $y(0) = y_0$
    (b) $\frac{dP}{dt} = rP(1 - \frac{P}{K})$ (logistic growth)
    (c) $\frac{dT}{dt} = -k(T - T_{\text{env}})$ (Newton's cooling)
    (d) $\frac{dN}{dt} = -\lambda N$ (radioactive decay)

39. A bacteria culture follows logistic growth: $P(t) = \frac{L}{1+ae^{-kt}}$
    (a) If $L = 2000$, $P(0) = 80$, and $P(1) = 160$, find a and k
    (b) Find the population after 8 days
    (c) When does the population reach 1000?
    (d) Find the maximum growth rate and when it occurs
    (e) Compare with exponential growth $P = 80e^{rt}$

40. The Arrhenius equation in chemistry is $k = Ae^{-E_a/(RT)}$ where k is reaction rate.

(a) Take natural logarithms to linearize the equation
(b) If at temperature 310K, $k = 0.012$, and at 360K, $k = 0.15$, find E_a/R
(c) Find the activation energy if $R = 8.314$ J/(mol·K)
(d) Predict the rate constant at 400K
(e) At what temperature does the rate increase 5-fold from 310K?

Section I: Advanced Applications and Modeling

41. A pharmacokinetic model describes drug concentration: $C(t) = \frac{D}{V}e^{-kt}$ where D is dose, V is volume of distribution, k is elimination rate.
    (a) If $D = 800$ mg, $V = 60$ L, $k = 0.15$ h^{-1}, find the initial concentration
    (b) Calculate the concentration after 6 hours
    (c) Find the half-life of the drug
    (d) When does concentration drop to 2.5 mg/L?
    (e) Model repeated dosing every 4 hours

42. Economic growth follows $Y(t) = Y_0e^{rt}$ where r is the growth rate.
    (a) If GDP grows at 6% per year, how long to double?
    (b) A country's GDP is £3 trillion and grows to £4.8 trillion in 6 years. Find the growth rate
    (c) Compare linear growth $Y = Y_0(1 + rt)$ with exponential over 35 years
    (d) Find when exponential growth overtakes linear with same initial rate
    (e) Model with varying growth rate $r(t) = r_0e^{-\delta t}$

43. The spread of an epidemic follows $I(t) = \frac{N}{1+(N/I_0-1)e^{-rt}}$ (logistic model).
    (a) If $N = 20000$, $I_0 = 25$, $r = 0.12$ per day, find infections after 16 days
    (b) When do infections peak?
    (c) Find the maximum rate of spread
    (d) Compare with exponential model $I = I_0e^{rt}$ for early stages
    (e) Model intervention reducing r by 70% after day 22

44. Weber-Fechner law relates stimulus and perception: $P = k\log(S/S_0)$.
    (a) If increasing stimulus 5-fold increases perception by 25 units, find k
    (b) Find perception when stimulus increases 16-fold
    (c) A sound's loudness follows $L = 10\log_{10}(I/I_0)$. Compare two sounds differing by 35 dB
    (d) Model brightness perception where threshold $S_0 = 0.12$ lux
    (e) Explain why equal ratios produce equal differences in perception

45. Design an optimization problem involving exponentials:
    (a) A company's profit is $P(t) = 2500e^{0.04t} - 800t$ over t years
    (b) Find when profit is maximized
    (c) Calculate maximum profit
    (d) Determine break-even points
    (e) Model with discounting: present value $= \frac{P(t)}{e^{rt}}$
    (f) Find optimal time to sell considering 8% discount rate

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

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