A Level Pure Mathematics
Practice Test 3: Differentiation

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Core Polynomial Derivatives

1. Differentiate these polynomial expressions:
   (a) $f(x) = 5x^6 - 4x^3 + 3x^2 - 6x + 4$
   (b) $g(x) = 2x^5 - \frac{3}{4}x^4 + x - 8$
   (c) $h(x) = (x-3)(x+5)$
   (d) $k(x) = (4x-1)^2$
   (e) $p(x) = x^4(x-2)$
   (f) $q(x) = \frac{x^5+x^3-4}{x}$

2. Find $\frac{dy}{dx}$ for these functions:
   (a) $y = 5x^{-4} - 2x^{-2} + 3$
   (b) $y = \frac{4}{x^4} + \frac{6}{\sqrt{x}} - 5\sqrt{x}$
   (c) $y = 4\sqrt{x^7} - \frac{5}{x^4} + x^{-\frac{2}{3}}$
   (d) $y = (3x + \frac{2}{x})^2$

3. Calculate gradients at the given points:
   (a) $y = x^5 - 3x^3 + 2x - 4$ at $x = 1$
   (b) $y = 2x^4 - x^3 + 5$ at $x = -1$
   (c) $y = \frac{x^4+2}{x}$ at $x = 2$
   (d) $y = (x-3)^3$ at $x = 4$

4. Determine tangent line equations for:
   (a) $y = x^5 - 2x^3 + x^2$ at the point where $x = 1$
   (b) $y = 4x^2 - 6x + 2$ at the point $(2, 2)$
   (c) $y = x^3 + 3x$ at the point where the gradient is 12
   (d) $y = \frac{x^4}{4} - x + 3$ at the point where $x = 2$

5. Given $f(x) = mx^3 + nx^2 + px + q$ with $f'(x) = 12x^2 + 6x - 3$:
   (a) Find the coefficients m, n, and p
   (b) If $f(0) = 7$, determine q
   (c) Write the complete function $f(x)$
   (d) Evaluate $f(2)$ and $f'(1)$

Section B: Transcendental Function Derivatives

6. Differentiate exponential and logarithmic functions:
   (a) $f(x) = 3e^x$
   (b) $g(x) = e^x + 4x^2$
   (c) $h(x) = x^3e^x$
   (d) $k(x) = 3\ln x$
   (e) $p(x) = x\ln x - x$
   (f) $q(x) = \frac{\ln x}{x^2}$
7. Find derivatives of trigonometric functions:
   (a) $f(x) = 3\sin x + 2\cos x$
   (b) $g(x) = 5\cos x - \sin x + x^2$
   (c) $h(x) = x\cos x$
   (d) $k(x) = \frac{\tan x}{x}$
   (e) $p(x) = \sec^2 x$
   (f) $q(x) = \sin x + \cos x$
8. Compute $\frac{dy}{dx}$ for:
   (a) $y = e^{4x}$
   (b) $y = \ln(4x)$
   (c) $y = \sin(5x)$
   (d) $y = \cos(2x - 3)$
   (e) $y = e^{x^4}$
   (f) $y = \ln(x^2 + 3)$
9. Apply mixed rules to differentiate:
   (a) $f(x) = e^x \tan x$
   (b) $g(x) = x^4 \sin x$
   (c) $h(x) = \frac{\ln x}{x}$
   (d) $k(x) = \frac{\tan x}{\cos x}$
   (e) $p(x) = (\ln x)^4$
   (f) $q(x) = \sqrt{\sin x + 1}$
10. Calculate derivatives of:
    (a) $f(x) = \sin^3 x$
    (b) $g(x) = \cos^5 x$
    (c) $h(x) = e^{\tan x}$
    (d) $k(x) = \ln(\cos x + 1)$
    (e) $p(x) = (\cos x + \sin x)^3$
    (f) $q(x) = \cos^{-1} x$ (inverse cosine)

Section C: Product and Quotient Rule Mastery

11. Use product rule for these functions:
    (a) $f(x) = (x^4 + 3)(x^2 - 1)$
    (b) $g(x) = (4x - 2)(x^4 + x^2 - 3)$
    (c) $h(x) = x^4 e^x$
    (d) $k(x) = (3x + 2)\ln x$
    (e) $p(x) = \sin x \tan x$
    (f) $q(x) = x^4 \sin x$
12. Apply quotient rule to find derivatives:
    (a) $f(x) = \frac{x^4+2}{x-3}$
    (b) $g(x) = \frac{4x-1}{x^3+2}$
    (c) $h(x) = \frac{\ln x}{x}$
    (d) $k(x) = \frac{x^2}{e^x+1}$
    (e) $p(x) = \frac{\tan x}{1+\cos x}$
    (f) $q(x) = \frac{x^4}{\sin x}$
13. Choose optimal method and differentiate:
    (a) $f(x) = \frac{x^4-5x^2}{x}$
    (b) $g(x) = (x^4 + 2)(x - 1)$
    (c) $h(x) = \frac{x^2-3x+2}{x^2}$
    (d) $k(x) = x^3(x^3 + 2)^2$
    (e) $p(x) = \frac{(3x+2)^2}{x}$
    (f) $q(x) = x^4\sqrt{x + 3}$
14. For $f(x) = x^4$ and $g(x) = \sin x$:
    (a) Find $(fg)'(x)$ using product rule
    (b) Calculate $(\frac{f}{g})'(x)$ using quotient rule
    (c) Evaluate $(fg)'(\frac{\pi}{6})$
    (d) Compute $(\frac{f}{g})'(\frac{\pi}{3})$
15. Demonstrate these rules:
    (a) Linear combination: $(af + bg)' = af' + bg'$
    (b) Power rule verification for x^n
    (c) Show that $(\frac{c}{v})' = -\frac{cv'}{v^2}$ for constant c
    (d) Verify that $(f^3)' = 3f^2 f'$

Section D: Chain Rule and Composite Functions

16. Apply chain rule to differentiate:
    (a) $f(x) = (4x + 3)^5$
    (b) $g(x) = (x^4 - 3x^2 + 2)^3$
    (c) $h(x) = \sqrt{x^4 + 2}$
    (d) $k(x) = (5x - 1)^{-4}$
    (e) $p(x) = \sin(4x + 2)$
    (f) $q(x) = \cos(x^4)$
17. Find $\frac{dy}{dx}$ for:
    (a) $y = e^{4x-2}$
    (b) $y = \ln(4x + 1)$
    (c) $y = (x^4 + 3x)^3$
    (d) $y = \sin^3 x$
    (e) $y = \cos(e^x)$
    (f) $y = e^{\sin x+1}$
18. Differentiate complex composites:
    (a) $f(x) = (e^x + 2)^4$
    (b) $g(x) = \ln(x^4 - x^2 + 1)$
    (c) $h(x) = \sin(\ln x + 1)$
    (d) $k(x) = e^{x \tan x}$
    (e) $p(x) = (\sin x + \cos x + 1)^5$
    (f) $q(x) = \ln(\sin x + 2)$
19. Use multiple differentiation rules:
    (a) $f(x) = x^3(4x - 1)^2$
    (b) $g(x) = \frac{x^3}{(3x+2)^4}$
    (c) $h(x) = x^2 \cos(4x)$
    (d) $k(x) = e^x \sin(4x)$
    (e) $p(x) = \frac{\ln x}{\sqrt{x+1}}$
    (f) $q(x) = \frac{(x^4+1)^3}{x^2}$
20. Compute second derivatives:
    (a) $f(x) = (3x + 1)^2$
    (b) $g(x) = \sin(4x)$
    (c) $h(x) = e^{-3x}$
    (d) $k(x) = \ln(x^4)$
    (e) $p(x) = x^3 e^x$
    (f) $q(x) = \cos x \sin x$

Section E: Stationary Points and Critical Analysis

21. Find coordinates of all stationary points:
    (a) $f(x) = x^3 + 3x^2 - 9x + 5$
    (b) $g(x) = 2x^3 - 12x^2 + 18x + 1$
    (c) $h(x) = x^4 - 6x^2 + 8$
    (d) $k(x) = \frac{x^4}{x-2}$ for $x \neq 2$

22. Use second derivative test to classify points:
    (a) $f(x) = x^3 - 12x^2 + 36x + 2$
    (b) $g(x) = 2x^3 - 6x^2 - 18x + 10$
    (c) $h(x) = x^4 - 8x^2 + 12$
    (d) $k(x) = xe^{-2x}$

23. Analyze all critical points completely:
    (a) $f(x) = x^3 - 6x^2 + 9x - 1$
    (b) $g(x) = 3x^3 - 18x^2 + 27x + 4$
    (c) $h(x) = x^4 - 12x^2 + 20$
    (d) $k(x) = x^3 - \frac{12}{x}$ for $x > 0$

24. For polynomial $f(x) = ax^3 + bx^2 + cx + d$:
    (a) State conditions for no stationary points
    (b) Show $f(x) = x^3 + 6x^2 + 12x + 8$ has no stationary points
    (c) Find k values so $f(x) = x^3 - 6kx^2 + 3$ has local minimum at $x = 4$

25. Study the function $f(x) = \frac{x^3-8}{x}$:
    (a) Find the domain
    (b) Calculate $f'(x)$ and locate stationary points
    (c) Determine nature of stationary points
    (d) Find all asymptotes
    (e) Sketch the graph completely

Section F: Motion and Rate Applications

26. A projectile's height is $h(t) = -5t^2 + 20t + 15$ meters after t seconds.
    (a) Find velocity $v(t)$ and acceleration $a(t)$
    (b) Determine when the projectile reaches maximum height
    (c) Calculate maximum height and time to hit ground
    (d) Find velocity when projectile hits ground
    (e) Calculate total distance traveled

27. A rectangular prism has square base with side x and height $h = 12 - x$. If base side increases at 2 cm/s:
    (a) Find rate of volume change when $x = 3$ cm
    (b) Express $\frac{dV}{dt}$ in terms of x and $\frac{dx}{dt}$

(c) When is volume decreasing at 24 cm^3/s?
(d) Find rate of surface area change when $x = 4$ cm

28. A 13-meter rope connects a boat to a dock 5 meters above water. The boat moves away at 3 m/s.

    (a) Set up the geometric relationship
    (b) Find how fast rope length increases when boat is 12m from dock base
    (c) Calculate rate of angle change with water
    (d) When does rope length increase fastest?

29. Water flows from a spherical tank of radius 6m at rate proportional to $\sqrt{h}$ where h is depth.

    (a) Express volume in terms of depth h
    (b) If $\frac{dV}{dt} = -2\sqrt{h}$, find $\frac{dh}{dt}$ when $h = 4$m
    (c) Calculate surface area change rate when $h = 9$m
    (d) When does water level drop fastest?

30. A bacterial culture grows according to $N(t) = 1000e^{0.05t}$ where t is hours.

    (a) Find growth rate $\frac{dN}{dt}$
    (b) Calculate population and growth rate after 4 hours
    (c) When is population growing at 100 bacteria per hour?
    (d) Express relative growth rate as percentage

Section G: Advanced Optimization

31. A gardener has 400m of fencing to enclose a rectangular garden with diagonal path (no fence on path).

    (a) Express area as function of one variable
    (b) Find dimensions for maximum area
    (c) Calculate maximum area
    (d) Verify using second derivative test

32. A rectangular container (no top) has volume 2000 cm^3. Bottom costs £6/m^2, sides cost £3/m^2.

    (a) Express cost in terms of base dimensions
    (b) Find dimensions for minimum cost
    (c) Calculate minimum cost
    (d) Find optimal height-to-width ratio

33. A tech company's daily output is $Q(x) = -x^3 + 12x^2 + 60x - 200$ units for x workers.

    (a) Find worker levels for maximum and minimum output
    (b) Calculate maximum daily output
    (c) Determine marginal productivity function
    (d) Find optimal workforce size

34. A church window is rectangular with semicircular top, total area 20 m^2.

    (a) Express perimeter in terms of rectangle width

(b) Find dimensions for minimum perimeter
(c) Calculate minimum perimeter
(d) Determine ratio of rectangle height to width

35. A right pyramid with square base fits inside sphere of radius 6 cm. Maximize pyramid volume.
 (a) Express volume in terms of pyramid height
 (b) Find critical points
 (c) Determine optimal height and base edge
 (d) Calculate maximum volume
 (e) Verify this gives maximum

Section H: Implicit Methods and Advanced Rates

36. Find $\frac{dy}{dx}$ using implicit differentiation:
 (a) $x^2 + y^2 = 36$
 (b) $x^2 + 3xy + y^2 = 12$
 (c) $x^3 + y^3 = 9xy$
 (d) $\sin(xy) = 2x + y$
 (e) $e^{xy} = x^3 - y$
 (f) $\ln(x^2 + y) = x + y$

37. Find tangent equations at specified points:
 (a) $x^2 + y^2 = 20$ at $(2, 4)$
 (b) $x^2 - xy + y^2 = 7$ at $(2, 1)$
 (c) $x^3 + y^3 = 16$ at $(2, 2)$
 (d) $xe^y = 6$ at $(3, \ln 2)$

38. Calculate $\frac{d^2y}{dx^2}$ implicitly:
 (a) $x^2 + y^2 = 9$
 (b) $xy = 6$
 (c) $x^2 - y^2 = 8$

39. Two planes depart simultaneously. Plane A flies east at 500 km/h, Plane B flies north at 400 km/h.
 (a) Express separation distance as function of time
 (b) Find separation rate after 2 hours
 (c) When are they separating at 640 km/h?
 (d) Calculate minimum separation distance

40. A gas balloon expands so volume increases at 50 cm^3/s. Find rate of change of:
 (a) Radius when $r = 6$ cm
 (b) Surface area when $r = 8$ cm
 (c) Diameter when volume is 4000 cm^3
 (d) Volume when surface area is 500π cm^2

Section I: Real-World Modeling and Design

41. A stadium entrance is a parabolic arch with base 30m and height 10m.
    (a) Find equation of the parabolic arch
    (b) Calculate arch height at 5m from center
    (c) Find width at height 8m
    (d) Determine optimal placement for maximum rectangular opening
42. A manufacturer produces widgets at cost $C(x) = 100 + 20x + 0.1x^2$ and sells at price $P(x) = 50 - 0.05x$.
    (a) Find profit function
    (b) Determine production level for maximum profit
    (c) Calculate maximum profit
    (d) Find break-even production levels
    (e) Analyze market dynamics
43. Medicine concentration in bloodstream follows $C(t) = \frac{20t}{t^2+4}$ mg/L after t hours.
    (a) Find when concentration peaks
    (b) Calculate peak concentration
    (c) Determine elimination rate at $t = 3$
    (d) When is concentration decreasing fastest?
    (e) Find effective therapeutic window
44. Design optimal cylindrical silo: volume 1500 m^3, hemispherical top, flat bottom.
    (a) Express surface area in terms of cylinder radius
    (b) Find dimensions for minimum surface area
    (c) Calculate minimum surface area
    (d) Compare with standard cylindrical design
45. For environmental project, minimize material in rectangular solar panel array totaling 200 m^2.
    (a) Express perimeter in terms of dimensions
    (b) Find dimensions for minimum perimeter
    (c) Calculate minimum perimeter
    (d) Consider practical installation constraints
    (e) Discuss environmental impact optimization
46. A suspension bridge cable forms parabola $y = ax^2 + bx + c$ with towers 200m apart, 50m high.
    (a) Find cable equation if lowest point is 10m above road
    (b) Calculate cable slope at towers
    (c) Find cable length (use calculus approximation)
    (d) Determine tension distribution along cable
    (e) Analyze structural engineering implications
47. Create optimization model for renewable energy system:
    (a) Define system parameters and constraints

(b) Formulate cost/efficiency objective function

(c) Apply calculus to find optimal configuration

(d) Verify solution meets all practical requirements

(e) Discuss scalability and implementation challenges

(f) Consider economic and environmental trade-offs

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

For more resources and practice materials, visit:
stepupmaths.co.uk