A Level Pure Mathematics
Practice Test 6: Differential Equations

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Advanced Theory

1. Explain these concepts:
   (a) Lipschitz continuity and solution uniqueness
   (b) Maximal intervals of existence
   (c) Blow-up phenomena in finite time
   (d) Invariant sets and first integrals
   (e) Hamiltonian systems and conservation
   (f) Poincaré maps and return times

2. Analyze equation types:
   (a) $\left(\frac{d^2y}{dx^2}\right)^4 + x\frac{dy}{dx} = y^3$
   (b) $\frac{d^7y}{dx^7} - 5\frac{d^5y}{dx^5} + 6y = e^{2x}$
   (c) $\sinh\left(\frac{dy}{dx}\right) - xy = 2$
   (d) $\frac{d^4y}{dx^4} + \left(\frac{d^2y}{dx^2}\right)^3 = x^4$
   (e) $\frac{dy}{dx} + y^{3/2} = x^{1/2}$ (nonlinear first-order)
   (f) $\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right)$ (variable coefficient PDE)

3. Verify solution families:
   (a) $y = C_1e^{5x} + C_2xe^{5x}$ satisfies $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 0$
   (b) $y = x^3(C_1 + C_2\ln x + C_3(\ln x)^2)$ satisfies $x^3\frac{d^3y}{dx^3} - 3x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} - 6y = 0$
   (c) $y = e^{-2x}\sin(3x)$ is a solution to $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$
   (d) $r^2 = x^2 + y^2$ gives implicit solutions to $x + y\frac{dy}{dx} = 0$

4. Form equations from curves:
   (a) $y = Ce^{-5x} + De^{2x} + x$ (exponential plus linear)
   (b) $y = e^{-2x}(C_1\cos(4x) + C_2\sin(4x))$ (damped oscillation)
   (c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (ellipses with fixed a, b)

   (d) $y = \frac{C}{x^3}$ (inverse cubic law)

5. Stability and bifurcations:
   (a) For $\frac{dy}{dx} = \mu y - y^3$, find equilibria for different μ
   (b) Identify bifurcation points and types
   (c) Sketch bifurcation diagram
   (d) Analyze hysteresis effects and stability switching

Section B: Integration Techniques

6. Direct integration:
   (a) $\frac{dy}{dx} = 10x^4 - 8x^3 + 2$
   (b) $\frac{dy}{dx} = 5e^{-3x} - 1$
   (c) $\frac{dy}{dx} = \frac{12}{6x+1}$
   (d) $\frac{dy}{dx} = \cot^2(2x) + 1$
   (e) $\frac{dy}{dx} = \frac{6x^5}{x^6+64}$
   (f) $\frac{dy}{dx} = (4x^3 + 2x)e^{x^4+x^2}$

7. Boundary conditions:
   (a) $\frac{dy}{dx} = 24x^3 - 18x$, $y(1) = 4$
   (b) $\frac{dy}{dx} = 8e^{3x}$, $y(0) = 6$
   (c) $\frac{dy}{dx} = \cos(5x)$, $y(\pi/10) = 4$
   (d) $\frac{dy}{dx} = \frac{8}{x+6}$, $y(0) = \ln 6$ (for $x > -6$)
   (e) $\frac{dy}{dx} = 5x\sqrt{x^2 + 16}$, $y(0) = 12$

8. Higher-order integration:
   (a) $\frac{d^2y}{dx^2} = 30x^4 - 18x$, $y(0) = 6$, $y'(0) = -5$
   (b) $\frac{d^2y}{dx^2} = 5e^{3x}$, $y(0) = 4$, $y'(0) = 1$
   (c) $\frac{d^3y}{dx^3} = 48x^3$, $y(0) = 2$, $y'(0) = 5$, $y''(0) = -4$
   (d) $\frac{d^2y}{dx^2} = \sin(5x)$, $y(0) = 3$, $y(\pi/10) = 2$

9. Physical motion:
   (a) Acceleration $a = 15t - 12$. Find $v(t)$ and $s(t)$ if $v(0) = 5$, $s(0) = -3$.
   (b) Ball thrown downward at 20 m/s from 150m. Find impact time and velocity.
   (c) Harmonic motion: $\frac{d^2x}{dt^2} = -49x$. Solve for $x(0) = 2$, $\dot{x}(0) = 7$.
   (d) Beam deflection with distributed load proportional to x^3.

10. Growth models:
   (a) Exponential: $\frac{dP}{dt} = 0.18P$. If $P(0) = 300$, find tripling time.
   (b) Decay: $\frac{dN}{dt} = -0.06N$. Find time for 75% reduction.
   (c) Investment: $\frac{dA}{dt} = 0.1A + 1000$. Solve if $A(0) = 8000$.
   (d) Autocatalytic: $\frac{dx}{dt} = kx(a - x)$ where a is total amount.

Section C: Variable Separation

11. Separable forms:
    (a) $\frac{dy}{dx} = 10xy^6$
    (b) $\frac{dy}{dx} = \frac{y^5}{x^4}$
    (c) $\frac{dy}{dx} = e^{5x-4y}$
    (d) $\frac{dy}{dx} = \frac{x^4 \cos x}{y^5}$
    (e) $\frac{dy}{dx} = \frac{\sin x}{\cos^2 y}$
    (f) $\frac{dy}{dx} = \frac{x^4 y}{x^5+243}$
12. Initial conditions:
    (a) $\frac{dy}{dx} = 8xy$, $y(0) = 5$
    (b) $\frac{dy}{dx} = \frac{5y}{x}$, $y(1) = 15$ (for $x > 0$)
    (c) $\frac{dy}{dx} = \frac{x^6}{y^5}$, $y(0) = 4$
    (d) $\frac{dy}{dx} = y(7-y)$, $y(0) = 2$
    (e) $\frac{dy}{dx} = \frac{5x}{\sqrt{36-y^2}}$, $y(0) = 0$
13. Complex separable:
    (a) $(25+y^2)\frac{dy}{dx} = 6xy$
    (b) $\frac{dy}{dx} = \frac{ye^{6x}}{x^6+1}$
    (c) $\cos^4 y \frac{dy}{dx} = \sin(4x)$
    (d) $\frac{dy}{dx} = \frac{x^5(1+y^2)}{y(1+x^6)}$
    (e) $y^4 \ln y \frac{dy}{dx} = x^5$
14. Applications:
    (a) Virus spread: $\frac{dI}{dt} = 0.2I$, $I(0) = 50$. When does $I = 1000$?
    (b) Nuclear waste: $\frac{dW}{dt} = -0.08W$. Find 90% decay time.
    (c) Thermal equilibrium: $\frac{dT}{dt} = -k(T-10)$. Object 120°C to 90°C in 4 minutes.
    (d) Gompertz growth: $\frac{dP}{dt} = rP\ln(\frac{K}{P})$ where K is limit.
15. Separability analysis:
    (a) $\frac{dy}{dx} = x^2y + xy^3$ (separable)
    (b) $\frac{dy}{dx} = x^2 + xy + y^3$ (not separable)
    (c) $\frac{dy}{dx} = \tan(x-2y)$ (not separable)
    (d) $\frac{dy}{dx} = e^{4x+3y}$ (separable)
    (e) $\frac{dy}{dx} = \frac{x^3y^3}{x^2+1}$ (separable)

Section D: Linear First-Order Systems

16. Integrating factor solutions:
    (a) $\frac{dy}{dx} + 8y = e^{7x}$
    (b) $\frac{dy}{dx} - 6y = 5x^5$
    (c) $\frac{dy}{dx} + \frac{7y}{x} = x^6$ (for $x > 0$)
    (d) $\frac{dy}{dx} + 3y\sin x = \cos x \sin x$
    (e) $x\frac{dy}{dx} + 7y = x^5$
    (f) $\frac{dy}{dx} + 6xy = 4xe^{-3x^2}$
17. Boundary problems:
    (a) $\frac{dy}{dx} + 7y = 21e^{5x}$, $y(0) = 4$
    (b) $\frac{dy}{dx} - 5y = 15x$, $y(0) = 3$
    (c) $\frac{dy}{dx} + 6y = 18$, $y(0) = 0$
    (d) $\frac{dy}{dx} + \frac{5y}{x} = 10x$, $y(1) = 8$ (for $x > 0$)
18. Advanced linear:
    (a) $\frac{dy}{dx} + 3y\csc x = \cot x \csc x$
    (b) $(x^2 + 16)\frac{dy}{dx} + 2xy = x^2 + 16$
    (c) $\frac{dy}{dx} + \frac{6y}{x^2+1} = \frac{6x}{x^2+1}$
    (d) $x^6\frac{dy}{dx} + 5x^5y = x^8$ (for $x > 0$)
19. Engineering systems:
    (a) Inductor circuit: $L\frac{di}{dt} + Ri = V_0e^{-t/\tau}$ with exponential decay.
    (b) Chemical reactor: 500L tank, reactant enters at 6 L/min (4 mol/L), exits at 6 L/min.
    (c) Pension fund: $\frac{dP}{dt} = 0.11P - 3000$ (11% growth, £3000 withdrawal).
    (d) Heat transfer: $mc\frac{dT}{dt} + hA(T - T_\infty) = Q(t)$ with time-varying heat source.
20. Solution verification:
    (a) Solve $\frac{dy}{dx} = 7xy + 7x$ by separation
    (b) Solve as linear: $\frac{dy}{dx} - 7xy = 7x$
    (c) Verify solution equivalence
    (d) Compare computational requirements

Section E: Second-Order Homogeneous

21. Characteristic methods:
    (a) $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 21y = 0$
    (b) $\frac{d^2y}{dx^2} + 16\frac{dy}{dx} + 64y = 0$
    (c) $\frac{d^2y}{dx^2} - 14\frac{dy}{dx} + 53y = 0$
    (d) $\frac{d^2y}{dx^2} + 100y = 0$
    (e) $\frac{d^2y}{dx^2} - 81y = 0$

(f) $\frac{d^2y}{dx^2} + 14\frac{dy}{dx} + 50y = 0$

22. Root classification:
    (a) $m^2 - 14m + 45 = 0$ (distinct real)
    (b) $m^2 + 18m + 81 = 0$ (repeated real)
    (c) $m^2 + 10m + 61 = 0$ (complex conjugate)
    (d) $m^2 - 144 = 0$ (distinct real)
    (e) $m^2 + 36 = 0$ (pure imaginary)
23. Boundary value problems:
    (a) $\frac{d^2y}{dx^2} - 15\frac{dy}{dx} + 50y = 0$, $y(0) = 5$, $y'(0) = 3$
    (b) $\frac{d^2y}{dx^2} + 14\frac{dy}{dx} + 49y = 0$, $y(0) = 3$, $y'(0) = -5$
    (c) $\frac{d^2y}{dx^2} + 64y = 0$, $y(0) = 0$, $y'(0) = 8$
    (d) $\frac{d^2y}{dx^2} - 16\frac{dy}{dx} + 65y = 0$, $y(0) = 4$, $y'(0) = 6$
24. System dynamics:
    (a) Exponential solutions and growth rates
    (b) Oscillatory behavior with frequency analysis
    (c) Damping ratios and stability margins
    (d) Phase relationships and mode shapes
25. Higher dimensions:
    (a) $\frac{d^3y}{dx^3} - 7\frac{d^2y}{dx^2} + 14\frac{dy}{dx} - 8y = 0$
    (b) $\frac{d^4y}{dx^4} - 2401y = 0$
    (c) General theory for nth order systems

Section F: Forced Second-Order

26. Particular solutions:
    (a) $\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 20y = 40$
    (b) $\frac{d^2y}{dx^2} + 64y = 192x^2$
    (c) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 42y = e^{7x}$
    (d) $\frac{d^2y}{dx^2} + 49y = \cos(6x)$
    (e) $\frac{d^2y}{dx^2} - 49y = 7e^{7x}$
    (f) $\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 36y = x^2 + 6$
27. Resonance phenomena:
    (a) $\frac{d^2y}{dx^2} + 64y = \sin(8x)$ (resonance)
    (b) $\frac{d^2y}{dx^2} - 13\frac{dy}{dx} + 42y = e^{6x}$ (resonance)
    (c) $\frac{d^2y}{dx^2} + 25y = \cos(5x)$ (resonance)
    (d) Amplitude growth and instability
28. Complete solutions:

(a) $\frac{d^2y}{dx^2} + 36y = 72$, $y(0) = 5$, $y'(0) = 0$
(b) $\frac{d^2y}{dx^2} - 36y = 72x$, $y(0) = 0$, $y'(0) = 6$
(c) $\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 25y = 25e^{-5x}$, $y(0) = 4$, $y'(0) = -8$

29. Trial function methods:
    (a) Polynomial, exponential, trigonometric forcing
    (b) Product rules and combination techniques
    (c) Resonance modification strategies
    (d) Superposition principle applications

30. Alternative approaches:
    (a) Variation of parameters: $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = \frac{e^{4x}}{x}$
    (b) Wronskian construction and integration
    (c) Green's function formulation

Section G: Physical Applications

31. Oscillating systems:
    (a) Mass-spring: $m\frac{d^2x}{dt^2} + kx = 0$ with $x(0) = 8$, $\dot{x}(0) = 0$, $m = 6$ kg, $k = 54$ N/m
    (b) Calculate resonant frequency and energy distribution
    (c) Torsional vibrations with polar moment of inertia
    (d) Multi-degree-of-freedom systems with coupling

32. Damped dynamics:
    (a) $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$ with $m = 1$, $c = 9$, $k = 14$ (overdamped)
    (b) Critical: $m = 1$, $c = 14$, $k = 49$ with $x(0) = 6$, $\dot{x}(0) = -7$
    (c) Underdamped: $m = 4$, $c = 12$, $k = 40$ with $x(0) = 5$, $\dot{x}(0) = 0$
    (d) Quality factor and bandwidth analysis

33. External excitation:
    (a) $\frac{d^2x}{dt^2} + 81x = 162\cos(8t)$ with zero initial conditions
    (b) Transfer function and frequency response
    (c) Resonance: $\frac{d^2x}{dt^2} + 64x = 128\cos(8t)$
    (d) Nonlinear resonance and jump phenomena

34. Circuit dynamics:
    (a) RLC series: $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V(t)$
    (b) Parameters: $L = 1.5$ H, $R = 12$, $C = 0.06$ F, $V = 36$ V
    (c) Transient analysis and steady-state response
    (d) Power dissipation and energy storage

35. Biological systems:
    (a) Population oscillations: $\frac{d^2P}{dt^2} + a\frac{dP}{dt} + bP = f(t)$
    (b) Neural dynamics and action potential propagation
    (c) Ecosystem stability and species interactions
    (d) Epidemiological models with time delays

Section H: Advanced Methods

36. Special equation types:
    - (a) Homogeneous: $\frac{dy}{dx} = \frac{6x+5y}{4x}$ (substitute $v = \frac{y}{x}$)
    - (b) Bernoulli: $\frac{dy}{dx} + 7y = 5xy^6$ (substitute $v = y^{1-n}$)
    - (c) Exact: $(8x^7 + 6y)dx + (6x + 10y)dy = 0$
    - (d) Clairaut: $y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$
37. Reduction techniques:
    - (a) $\frac{d^2y}{dx^2} + \frac{6}{x}\frac{dy}{dx} = 0$ (substitute $v = \frac{dy}{dx}$)
    - (b) $y\frac{d^2y}{dx^2} = 6\left(\frac{dy}{dx}\right)^2$
    - (c) Euler: $x^2\frac{d^2y}{dx^2} + 9x\frac{dy}{dx} + 16y = 0$
    - (d) Phase plane reduction for autonomous systems
38. Matrix systems:
    - (a) $\frac{dx}{dt} = 7x + 5y$, $\frac{dy}{dt} = 5x + 7y$
    - (b) Eigenvalue decomposition and modal analysis
    - (c) Spiral, node, and saddle point classification
    - (d) Lyapunov stability and basin boundaries
39. Transform techniques:
    - (a) Laplace: $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 10y = H(t)$ (Heaviside function)
    - (b) Transfer functions and system identification
    - (c) Fourier series for periodic forcing
    - (d) Z-transforms for discrete systems
40. Boundary problems:
    - (a) $\frac{d^2y}{dx^2} + \lambda^2 y = 0$ with $y(0) = y(2\pi) = 0$
    - (b) Sturm-Liouville eigenvalue problems
    - (c) Green's functions and integral equations
    - (d) Variational principles and weak solutions

Section I: Capstone Project

41. Select comprehensive modeling project:
    - (a) Pandemic dynamics with behavioral responses
    - (b) Climate tipping cascades and feedback loops
    - (c) Financial contagion and systemic risk
    - (d) Neuronal networks and brain dynamics
    - (e) Quantum mechanical tunneling effects
    - (f) Aerospace trajectory optimization

    Complete investigation:
    - (a) Literature review and problem formulation

(b) Mathematical model development and validation
(c) Scaling analysis and dimensionless parameters
(d) Analytical approximations and exact solutions
(e) Numerical simulation and computational methods
(f) Parameter estimation and uncertainty quantification
(g) Sensitivity analysis and robustness testing
(h) Model comparison and selection criteria
(i) Policy implications and decision support
(j) Future research directions and open questions

42. Advanced numerics:

(a) Adaptive methods for $\frac{dy}{dx} = \lambda(y^2 - 1)$, $y(0) = 0$
(b) Stiff system solvers and implicit methods
(c) Shooting and collocation for boundary problems
(d) Spectral methods and pseudospectral approximation

43. Mathematical theory:

(a) Dynamical systems and chaos theory
(b) Bifurcation analysis and normal forms
(c) Perturbation methods and asymptotic analysis
(d) Homogenization and multiple scale techniques

44. Integration and synthesis:

(a) Unified classification framework
(b) Computational complexity and algorithm selection
(c) Historical evolution and future trends
(d) Interdisciplinary connections and applications
(e) Research methodology and scientific computing

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 250

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