A Level Pure Mathematics
Practice Test 2: Vectors

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Vector Basics and Notation

1. Given vectors $\mathbf{u} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$, calculate:

   (a) $\mathbf{u} + \mathbf{v}$
   (b) $\mathbf{u} - \mathbf{v}$
   (c) $3\mathbf{u} + 2\mathbf{v}$
   (d) $4\mathbf{u} - 3\mathbf{v}$
   (e) $|\mathbf{u}|$ and $|\mathbf{v}|$
   (f) A unit vector in the direction of $\mathbf{v}$

2. Express these vectors in component form:

   (a) $\overrightarrow{CD}$ where $C(3, 1, -2)$ and $D(1, 4, 3)$
   (b) $\overrightarrow{RS}$ where $R(-2, 3, 1)$ and $S(3, -1, 4)$
   (c) The position vector of point E if $\overrightarrow{OE} = 2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$
   (d) $\overrightarrow{DC}$ where $C(2, -3, 1)$ and $D(5, 2, -4)$

3. Given $\mathbf{m} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{n} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$:

   (a) Find $|\mathbf{m}|$ and $|\mathbf{n}|$
   (b) Calculate $\mathbf{m} + \mathbf{n}$ and $\mathbf{m} - \mathbf{n}$
   (c) Find scalars p and q such that $p\mathbf{m} + q\mathbf{n} = \begin{pmatrix} 1 \\ -4 \\ 6 \end{pmatrix}$
   (d) Determine if $\mathbf{m}$ and $\mathbf{n}$ are parallel

4. Points P, Q, and R have position vectors $\mathbf{p} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$, and $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$.

   (a) Find vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$
   (b) Calculate the lengths $|PQ|$ and $|PR|$

(c) Find the position vector of the midpoint of QR

(d) Determine if triangle PQR is isosceles

5. Find the values of k for which these vectors are perpendicular:

(a) $\mathbf{x} = \begin{pmatrix} 3 \\ k \\ 2 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} k \\ 1 \\ -3 \end{pmatrix}$

(b) $\mathbf{p} = \begin{pmatrix} 2 \\ 3k \\ 1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 1 \\ -2 \\ k \end{pmatrix}$

(c) $\mathbf{r} = k\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{s} = 2\mathbf{i} + k\mathbf{j} + 5\mathbf{k}$

Section B: Dot Product (Scalar Product)

6. Calculate the dot product of these vectors:

(a) $\mathbf{a} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$

(b) $\mathbf{p} = 2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ and $\mathbf{q} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

(c) $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$

(d) $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{s} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$

7. Find the angle between these pairs of vectors:

(a) $\mathbf{a} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(b) $\mathbf{p} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

(c) $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

(d) $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

8. Use the dot product to verify these properties:

(a) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (commutative)

(b) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ (distributive)

(c) $(k\mathbf{a}) \cdot \mathbf{b} = k(\mathbf{a} \cdot \mathbf{b})$ for scalar k

(d) $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

9. Given vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$:

(a) Show that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular

(b) Find the component of $\mathbf{c}$ in the direction of $\mathbf{a}$

(c) Calculate $|\mathbf{a} + \mathbf{b} + \mathbf{c}|$

(d) Find the angle between $\mathbf{a} + \mathbf{b}$ and $\mathbf{c}$

10. A triangle has vertices at $P(2, 1, 3)$, $Q(1, 4, 2)$, and $R(3, 2, 1)$.

    (a) Find the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$
    (b) Calculate the angle $\angle QPR$
    (c) Find the area of triangle PQR
    (d) Determine if the triangle is right-angled

Section C: Cross Product (Vector Product)

11. Calculate the cross product of these vectors:

    (a) $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$
    (b) $\mathbf{p} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{q} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$
    (c) $\mathbf{u} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$
    (d) $\mathbf{r} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{s} = \mathbf{i} + 3\mathbf{k}$

12. Verify these properties of the cross product:

    (a) $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$ (anti-commutative)
    (b) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (distributive)
    (c) $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
    (d) $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$

13. Find the area of the parallelogram spanned by:

    (a) $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$
    (b) $\mathbf{p} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{q} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$
    (c) Vectors from origin to points $(3, 1, 2)$ and $(2, 3, 1)$
    (d) $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ where $P(2, 1, 0)$, $Q(1, 3, 2)$, $R(3, 0, 1)$

14. Given $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$:

    (a) Calculate $\mathbf{a} \times \mathbf{b}$
    (b) Verify that $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$
    (c) Find a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$
    (d) Calculate the area of triangle with sides $\mathbf{a}$ and $\mathbf{b}$

15. Use the scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ to find:

    (a) The volume of parallelepiped with edges $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$
    (b) Whether points $P(2, 1, 3)$, $Q(1, 3, 2)$, $R(3, 2, 1)$, $S(2, 2, 2)$ are coplanar
    (c) The volume of tetrahedron with vertices at $(0, 0, 0)$, $(2, 1, 3)$, $(1, 2, 1)$, $(3, 1, 2)$

Section D: Equations of Lines

16. Find the vector equation of the line:

    (a) Passing through $P(1,3,2)$ in direction $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

    (b) Passing through points $A(2,1,4)$ and $B(1,3,2)$

    (c) Through origin parallel to vector $3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$

    (d) Through $(3,2,1)$ parallel to the line $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$

17. Convert these to parametric form:

    (a) $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$

    (b) $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

    (c) Line through $(2,3,1)$ and $(1,0,4)$

    (d) $\mathbf{r} = (2+3t)\mathbf{i} + (1-2t)\mathbf{j} + (4+t)\mathbf{k}$

18. Find where these lines intersect the coordinate planes:

    (a) $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ and the xy-plane

    (b) $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + s\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and the xz-plane

    (c) Line through $(3,2,1)$ and $(1,4,0)$ with the yz-plane

19. Determine if these pairs of lines intersect, are parallel, or are skew:

    (a) $L_1 : \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

    (b) $L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + s\begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}$

    (c) Lines through $(2,1,3)$ to $(4,2,1)$ and $(1,3,2)$ to $(3,1,4)$

20. Find the shortest distance between:

    (a) Point $(1,3,2)$ and line $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

    (b) Parallel lines $L_1 : \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

    (c) Skew lines $L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

Section E: Equations of Planes

21. Find the equation of the plane:

    (a) With normal vector $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ passing through $(2, 1, 4)$

    (b) Passing through points $(2, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 1)$

    (c) Containing the lines $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} + s\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

    (d) Parallel to vectors $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ through $(2, 2, 2)$

22. Convert between vector and Cartesian forms:

    (a) $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ to Cartesian form

    (b) $3x - 2y + z = 8$ to vector form

    (c) $2x + y - 3z = 6$ to parametric form

    (d) $\mathbf{r} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 7$ to Cartesian form

23. Find where these planes intersect coordinate axes:

    (a) $2x + 3y - z = 12$

    (b) $x - 3y + 2z = 6$

    (c) $3x + y + 2z = 18$

    (d) $2x + 2y + z = 6$

24. Determine the relationship between these planes:

    (a) $\Pi_1 : 2x + y - z = 4$ and $\Pi_2 : 4x + 2y - 2z = 8$

    (b) $\Pi_1 : x - 2y + 3z = 5$ and $\Pi_2 : 2x + y - z = 3$

    (c) $\Pi_1 : 3x + y + z = 6$ and $\Pi_2 : 6x + 2y + 2z = 15$

    (d) $\Pi_1 : 2x - y + 3z = 7$ and $\Pi_2 : x + 2y - z = 4$

25. Find the line of intersection of these planes:

    (a) $2x + y + z = 5$ and $x - y + 2z = 3$

    (b) $3x + y - 2z = 6$ and $x - 3y + z = 2$

    (c) $2x - 3y + z = 4$ and $x + 2y - 3z = 1$

    (d) $x + 3y + 2z = 9$ and $3x - y + z = 7$

Section F: Angles and Distances

26. Find the angle between these planes:

    (a) $2x + y - 3z = 4$ and $x - 2y + z = 6$

    (b) $3x + 2y - z = 8$ and $2x - 3y + 2z = 5$

(c) $\mathbf{r} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 3$ and $\mathbf{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 4$

(d) $3x + y + 2z = 9$ and $x - 2y + 3z = 6$

27. Calculate the distance from point to plane:

(a) Point $(1, 3, 2)$ to plane $2x + y - 3z = 4$

(b) Point $(3, -1, 4)$ to plane $x - 2y + 2z = 8$

(c) Point $(0, 0, 0)$ to plane $3x + 2y - z = 15$

(d) Point $(2, 4, -1)$ to plane $2x - 3y + z = 7$

28. Find the angle between line and plane:

(a) Line $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and plane $2x + y + z = 8$

(b) Line through $(2, 1, 3)$ and $(1, 4, 2)$ with plane $3x - y + 2z = 6$

(c) Line $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and plane $x + 2y - z = 4$

29. Determine where these lines intersect planes:

(a) $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $x + 2y + z = 12$

(b) Line through $(2, 3, 1)$ and $(1, 1, 4)$ with plane $3x - y + 2z = 8$

(c) $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + s \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ and $2x - y + 3z = 15$

30. Find the reflection of point in plane:

(a) Point $(3, 1, 2)$ in plane $x + 2y - z = 4$

(b) Point $(2, -1, 3)$ in plane $3x - y + 2z = 8$

(c) Point $(1, 4, 0)$ in plane $2x + y + 3z = 6$

Section G: Advanced Vector Geometry

31. A tetrahedron has vertices at $P(2, 0, 0)$, $Q(0, 3, 0)$, $R(0, 0, 4)$, and $S(2, 2, 2)$.

(a) Find the volume of the tetrahedron

(b) Calculate the area of face PQR

(c) Find the equation of the plane containing face PQR

(d) Determine the perpendicular distance from S to plane PQR

(e) Verify the volume using the distance formula

32. Three forces $\mathbf{F_1} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, $\mathbf{F_2} = \mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$, and $\mathbf{F_3} = -3\mathbf{i} + \mathbf{j} + \mathbf{k}$ act on a particle.

(a) Find the resultant force

(b) Calculate the magnitude of the resultant

(c) Find a fourth force needed for equilibrium

(d) If the forces act at point $(2, 1, 3)$, find the moment about the origin

33. A regular tetrahedron has vertices at $(2, 2, 2)$, $(2, -2, -2)$, $(-2, 2, -2)$, and $(-2, -2, 2)$.

    (a) Verify that all edges have equal length
    (b) Find the center of the tetrahedron
    (c) Calculate the angle between any two faces
    (d) Find the equation of the sphere circumscribing the tetrahedron

34. The position vectors of points A, B, and C are $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ respectively.

    (a) Express the centroid G in terms of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$
    (b) Show that $\overrightarrow{AG} + \overrightarrow{BG} + \overrightarrow{CG} = \mathbf{0}$
    (c) If M is the midpoint of BC, express $\overrightarrow{AM}$ in terms of position vectors
    (d) Prove that the medians of triangle ABC meet at the centroid

35. A line passes through point $P(1, 3, 2)$ and is perpendicular to the plane $3x - y + 2z = 8$.

    (a) Find the vector equation of the line
    (b) Calculate where the line intersects the plane
    (c) Find the foot of perpendicular from P to the plane
    (d) Calculate the distance from P to the plane

Section H: Applications and Problem Solving

36. A parallelogram $PQRS$ has vertices $P(2, 1, 3)$, $Q(1, 4, 2)$, and $R(3, 2, 5)$.

    (a) Find the coordinates of vertex S
    (b) Calculate the area of the parallelogram
    (c) Find the lengths of the diagonals
    (d) Determine if the parallelogram is a rhombus
    (e) Calculate the angle between the diagonals

37. A spacecraft travels from station $P(200, 150, 10)$ to station $Q(500, 300, 15)$ (coordinates in km).

    (a) Find the displacement vector $\overrightarrow{PQ}$
    (b) Calculate the distance traveled
    (c) If the journey takes 3 hours, find the average velocity vector
    (d) Find the bearing of Q from P (projected onto horizontal plane)
    (e) Calculate the angle of ascent

38. A pyramid has square base with vertices at $(3, 3, 0)$, $(-3, 3, 0)$, $(-3, -3, 0)$, $(3, -3, 0)$ and apex at $(0, 0, 6)$.

    (a) Find the volume of the pyramid
    (b) Calculate the area of each triangular face
    (c) Find the total surface area
    (d) Determine the angle between a triangular face and the base
    (e) Find the equation of the plane containing one triangular face

39. Two particles move along lines $L_1 : \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + s \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

    (a) Show that the lines are skew
    (b) Find the shortest distance between the lines
    (c) If the particles start at $t = s = 0$ and move with constant speeds, when are they closest?
    (d) Calculate their closest approach distance
    (e) Find the common perpendicular to both lines

40. A sphere has center $O(1, 2, -1)$ and radius 4.

    (a) Write the equation of the sphere
    (b) Find where the sphere intersects the plane $x + 2y + z = 6$
    (c) Determine if the line $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ intersects the sphere
    (d) Find the equation of the tangent plane at point $(5, 2, -1)$
    (e) Calculate the volume and surface area of the sphere

Section I: Advanced Topics and Modeling

41. Prove these vector identities:

    (a) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$ (vector triple product)
    (b) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$
    (c) $|\mathbf{a} \times \mathbf{b}|^2 + |\mathbf{a} \cdot \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2$ (Lagrange identity)
    (d) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$

42. Three planes $\Pi_1 : 2x + y - z = 4$, $\Pi_2 : x - 2y + 3z = 6$, and $\Pi_3 : 3x + y + 2z = 9$ intersect.

    (a) Find their common point of intersection
    (b) Calculate the angles between each pair of planes
    (c) Find the line of intersection of Π_1 and Π_2
    (d) Determine the volume of the tetrahedron formed by the three planes and the origin
    (e) Verify the intersection point lies on all three planes

43. A coordinate system undergoes rotation. The new basis vectors are: $\mathbf{e_1'} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{e_2'} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{e_3'} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

    (a) Verify these form an orthonormal basis
    (b) Express vector $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ in the new coordinate system
    (c) Find the rotation matrix for this transformation
    (d) Calculate the axis and angle of rotation

44. A crystal has lattice vectors $\mathbf{a} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1.5 \\ 1.5\sqrt{3} \\ 0 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$.

 (a) Calculate the volume of the unit cell
 (b) Find the angles between the lattice vectors
 (c) Determine the reciprocal lattice vectors
 (d) Calculate the density if each unit cell contains 6 atoms of mass 1.5×10^{-23} g
 (e) Find the distance between parallel planes with Miller indices $(2, 1, 0)$

45. Design a vector-based model for a real-world application:

 (a) Choose a scenario involving 3D geometry (navigation, animation, structural analysis)
 (b) Define your coordinate system and relevant vectors clearly
 (c) Set up vector equations describing the system
 (d) Solve a specific problem using vector methods
 (e) Discuss advantages of vector methods for your application
 (f) Consider limitations and potential extensions of your model

46. A satellite orbits Earth in an elliptical path. At time t, its position vector is: $\mathbf{r}(t) = 6\cos(\omega t)\mathbf{i} + 4\sin(\omega t)\mathbf{j} + 2\mathbf{k}$

 (a) Find the velocity vector $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$
 (b) Calculate the acceleration vector $\mathbf{a}(t) = \frac{d\mathbf{v}}{dt}$
 (c) Show that acceleration is always directed toward a point in the xy-plane
 (d) Find the speed as a function of time
 (e) Determine when the satellite is closest to the xy-plane
 (f) Calculate the angular momentum $\mathbf{L} = \mathbf{r} \times m\mathbf{v}$

47. Vector calculus in 3D space involves del operator $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$.

 (a) For scalar field $\phi(x, y, z) = x^2 y + z^3$, find $\nabla\phi$
 (b) For vector field $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$, calculate $\nabla \cdot \mathbf{F}$ (divergence)
 (c) Find $\nabla \times \mathbf{F}$ (curl) for the same vector field
 (d) Verify that $\nabla \times (\nabla\phi) = \mathbf{0}$ for any scalar field ϕ
 (e) Show that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ for any vector field $\mathbf{F}$

48. Integration and applications:

 (a) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ and C is the curve from $(0, 0)$ to $(2, 1)$ along $y = \frac{x^2}{4}$
 (b) Calculate the work done by force $\mathbf{F} = (y + z)\mathbf{i} + (x - z)\mathbf{j} + (x + y)\mathbf{k}$ moving a particle from $(0, 0, 0)$ to $(2, 1, 3)$
 (c) Find the flux of $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ through the surface of unit sphere
 (d) Apply Green's theorem to evaluate $\oint_C (x^2 - y^2)dx + 2xy\,dy$ around a circle of radius 2

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

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