A Level Pure Mathematics
Practice Test 4: Sequences and Series

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Arithmetic Sequences

1. For the arithmetic sequence $11, 18, 25, 32, 39, \ldots$:
   (a) Find the first term a and common difference d
   (b) Find the general term u_n
   (c) Calculate u_{28}
   (d) Find which term equals 158
   (e) Determine if 300 is a term in the sequence

2. An arithmetic sequence has $u_6 = 38$ and $u_{12} = 68$.
   (a) Find the first term and common difference
   (b) Write the general term u_n
   (c) Calculate u_{22}
   (d) Find the first term to exceed 150
   (e) Determine the largest value of n for which $u_n < 180$

3. The nth term of an arithmetic sequence is $u_n = 6n - 5$.
   (a) Write down the first five terms
   (b) Find the common difference
   (c) Calculate u_{45}
   (d) Find the sum of the first 35 terms
   (e) For what value of n is $u_n = 175$?

4. Three numbers $q-4j$, q, and $q+4j$ are in arithmetic progression with sum 57 and product 5187.
   (a) Find the value of q
   (b) Set up an equation for j
   (c) Solve to find the values of j
   (d) Write down the three numbers for each case

5. An arithmetic sequence has first term a and common difference d.
   (a) If the mth term is x times the nth term, prove that $(mx - n)a = (m - nx)d$
   (b) Show that if $S_m = S_n$ where $m \neq n$, then $S_{m+n} = 0$
   (c) Prove that the arithmetic mean of the first n terms equals $\frac{u_1+u_n}{2}$
   (d) If consecutive terms u_p, u_q, u_r satisfy $u_p + u_r = 2u_q$, show that $p + r = 2q$

Section B: Arithmetic Series

6. Calculate the sum of these arithmetic series:
   (a) $8 + 14 + 20 + 26 + \ldots$ (first 20 terms)
   (b) $35 + 31 + 27 + 23 + \ldots$ (first 18 terms)
   (c) $\frac{2}{5} + \frac{4}{5} + \frac{6}{5} + \frac{8}{5} + \ldots$ (first 24 terms)
   (d) The series with first term 18, last term 126, and 13 terms
7. An arithmetic series has first term 11 and common difference 7.
   (a) Find the sum of the first 16 terms
   (b) Find the smallest value of n for which $S_n \geq 2500$
   (c) If the sum of the first n terms is 1680, find n
   (d) Express S_n in terms of n
8. The sum of the first n terms of an arithmetic series is $S_n = 4n^2 - 2n$.
   (a) Find the first term u_1
   (b) Find u_2 and u_3
   (c) Determine the common difference
   (d) Find the general term u_n
   (e) Verify using the formula $u_n = S_n - S_{n-1}$ for $n \geq 2$
9. Find the sum of:
   (a) All multiples of 8 between 150 and 600
   (b) All integers from 1 to 80 that are divisible by 6
   (c) All odd integers from 11 to 199
   (d) The integers from 1 to 200 that are divisible by 4 or 5
10. An arithmetic series has $S_{14} = 406$ and $S_{22} = 946$.
    (a) Find the first term and common difference
    (b) Calculate S_{30}
    (c) Find the 18th term
    (d) Determine when the sum first exceeds 2500

Section C: Geometric Sequences

11. For the geometric sequence $5, 20, 80, 320, 1280, \ldots$:
    (a) Find the first term a and common ratio r
    (b) Find the general term u_n
    (c) Calculate u_{11}
    (d) Find which term equals 5120
    (e) Determine if 20480 is a term in the sequence
12. A geometric sequence has $u_5 = 48$ and $u_8 = 384$.
    (a) Find the common ratio r
    (b) Find the first term a

(c) Write the general term u_n
(d) Calculate u_{13}
(e) Find the first term to exceed 200000

13. The nth term of a geometric sequence is $u_n = 8 \times 2^{n-1}$.
    (a) Write down the first five terms
    (b) Find the common ratio
    (c) Calculate u_{15}
    (d) Find the sum of the first 9 terms
    (e) For what value of n is $u_n = 2048$?

14. Three numbers $\frac{v}{u}$, v, and vu are in geometric progression with sum 124 and product 1728.
    (a) Find the value of v
    (b) Set up an equation for u
    (c) Solve to find the values of u
    (d) Write down the three numbers for each case

15. A geometric sequence has first term a and common ratio r.
    (a) If the geometric mean of u_m and u_n is u_k, prove that $2k = m + n$
    (b) Show that the sequence $u_1^2, u_2^2, u_3^2, \ldots$ is also geometric with ratio r^2
    (c) Prove that for any three consecutive terms, $u_{n-1} \cdot u_{n+1} = u_n^2$
    (d) If S_n denotes the sum of the first n terms, show that $S_n = a \cdot \frac{r^n-1}{r-1}$ when $r \neq 1$

Section D: Geometric Series

16. Calculate the sum of these geometric series:
    (a) $9 + 27 + 81 + 243 + \ldots$ (first 10 terms)
    (b) $3 - 12 + 48 - 192 + \ldots$ (first 11 terms)
    (c) $\frac{2}{5} + \frac{2}{15} + \frac{2}{45} + \frac{2}{135} + \ldots$ (first 12 terms)
    (d) $96 + 72 + 54 + 40.5 + \ldots$ (first 14 terms)

17. A geometric series has first term 15 and common ratio $\frac{3}{5}$.
    (a) Find the sum of the first 20 terms
    (b) Find the smallest value of n for which $S_n \geq 37$
    (c) Calculate the sum to infinity
    (d) Find how many terms are needed for the sum to be within 0.01 of the sum to infinity

18. The sum of the first n terms of a geometric series is $S_n = 6(5^n - 1)$.
    (a) Find the first term u_1
    (b) Find u_2 and u_3
    (c) Determine the common ratio
    (d) Find the general term u_n
    (e) Verify using the formula $u_n = S_n - S_{n-1}$ for $n \geq 2$

19. Evaluate these infinite geometric series:

(a) $1 + \frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \ldots$
(b) $8 - 4 + 2 - 1 + \ldots$
(c) $\frac{7}{8} + \frac{7}{32} + \frac{7}{128} + \frac{7}{512} + \ldots$
(d) $0.3 + 0.03 + 0.003 + 0.0003 + \ldots$

20. A geometric series has $S_6 = 126$ and $S_{12} = 4662$.
    (a) Set up equations for the first term and common ratio
    (b) Solve to find a and r
    (c) Calculate S_{18}
    (d) Find the sum to infinity (if it exists)
    (e) Determine the first term to exceed 10000

Section E: Sigma Notation

21. Evaluate these sums:
    (a) $\sum_{r=1}^{18}(5r+3)$
    (b) $\sum_{r=1}^{35}(6r-5)$
    (c) $\sum_{r=1}^{28} r^2$
    (d) $\sum_{r=1}^{16}(4r^2+3r)$

22. Express these series using sigma notation:
    (a) $10 + 16 + 22 + 28 + \ldots + 58$
    (b) $6 + 30 + 150 + 750 + \ldots + 18750$
    (c) $2^3 + 4^3 + 6^3 + 8^3 + \ldots + 20^3$
    (d) $\frac{1}{5} + \frac{1}{14} + \frac{1}{27} + \frac{1}{44} + \ldots + \frac{1}{104}$

23. Use the standard formulae to evaluate:
    (a) $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$: Find $\sum_{r=1}^{85} r$
    (b) $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$: Find $\sum_{r=1}^{35} r^2$
    (c) $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$: Find $\sum_{r=1}^{20} r^3$
    (d) $\sum_{r=1}^{45}(5r^2-4r+3)$

24. Simplify these expressions:
    (a) $\sum_{r=1}^{n}(mr+s)$ in terms of m, s, and n
    (b) $\sum_{r=1}^{n}(4r^2-2r+3)$
    (c) $\sum_{r=1}^{n}(3r+1)^2$
    (d) $\sum_{r=1}^{n} r(4r+1)$

25. Prove these results:
    (a) $\sum_{r=1}^{n}(5r-4) = \frac{n(5n-3)}{2}$
    (b) $\sum_{r=1}^{n} r(r+4) = \frac{n(n+1)(n+11)}{3}$
    (c) $\sum_{r=1}^{n} \frac{1}{(4r-3)(4r+1)} = \frac{n}{4n+1}$
    (d) $\sum_{r=1}^{n}((r+2)^2-(r+1)^2) = n(n+5)$

Section F: Binomial Expansion - Integer Powers

26. Expand using the binomial theorem:
   (a) $(x+5)^5$
   (b) $(5x-4)^4$
   (c) $(4-3x)^6$
   (d) $(4x+\frac{3}{x})^5$
27. Find the specified terms in these expansions:
   (a) The coefficient of x^6 in $(5x+3)^{10}$
   (b) The coefficient of x^8 in $(3x-2)^{11}$
   (c) The constant term in $(x^5+\frac{4}{x^3})^6$
   (d) The coefficient of x^{-2} in $(5x^3-\frac{2}{x})^7$
28. Use the binomial theorem to evaluate:
   (a) $(1.05)^6$ to 6 decimal places
   (b) $(0.94)^5$ to 5 decimal places
   (c) $(1.02)^8$ exactly
   (d) 103^4 by writing it as $(100+3)^4$
29. In the expansion of $(1+dx)^q$:
   (a) The coefficient of x is 21 and the coefficient of x^2 is 189. Find d and q.
   (b) Find the coefficient of x^3
   (c) Write out the first four terms of the expansion
   (d) For what values of x does the expansion converge?
30. The coefficient of x^i in the expansion of $(1+x)^q$ is $\binom{q}{i}$.
   (a) Show that $\sum_{i=0}^{q}\binom{q}{i}\cdot 3^i = 4^q$
   (b) Prove that $\binom{q}{i}\cdot\binom{i}{j} = \binom{q}{j}\cdot\binom{q-j}{i-j}$ for appropriate values
   (c) Use Pascal's triangle properties to find $\binom{12}{5}$ from known values
   (d) Show that $\sum_{i=0}^{q}(-1)^i\binom{q}{i}\cdot i^2 = 0$ for $q \geq 2$

Section G: Binomial Expansion - Non-Integer Powers

31. Expand these expressions up to and including the term in x^3:
   (a) $(1+x)^{3/4}$
   (b) $(1-x)^{-4}$
   (c) $(1+6x)^{1/2}$
   (d) $(1-7x)^{-1/3}$
32. Find the first four terms in the expansion of:
   (a) $(36+x)^{1/2}$
   (b) $(25-x)^{-1/2}$
   (c) $\frac{1}{(4+x)^4}$
   (d) $\sqrt{16-5x}$

33. State the range of values of x for which these expansions are valid:
   (a) $(1+6x)^{-1} = 1 - 6x + 36x^2 - 216x^3 + \dots$
   (b) $(1-5x)^{1/2} = 1 - \frac{5x}{2} - \frac{25x^2}{8} - \frac{125x^3}{16} - \dots$
   (c) $(7+x)^{-1} = \frac{1}{7} - \frac{x}{49} + \frac{x^2}{343} - \frac{x^3}{2401} + \dots$
   (d) $\frac{1}{\sqrt{25-x}} = \frac{1}{5} + \frac{x}{250} + \frac{3x^2}{12500} + \dots$
34. Use binomial expansions to find approximations:
   (a) $\sqrt{1.08}$ to 5 decimal places
   (b) $\frac{1}{\sqrt{0.88}}$ to 4 decimal places
   (c) $(1.05)^{-5}$ to 6 decimal places
   (d) $\sqrt[3]{1.12}$ to 5 decimal places
35. Find the coefficient of x^2 in the expansion of:
   (a) $(1+x)^{3/4}(1-x)^{1/4}$
   (b) $(1+5x)^{-1}(1+3x)^2$
   (c) $\frac{1+3x}{\sqrt{1-x}}$
   (d) $(1+2x-x^2)(1+x)^{-4}$

Section H: Mixed Series and Advanced Topics

36. A sequence is defined by $u_1 = 5$ and $u_{n+1} = 4u_n - 11$ for $n \geq 1$.
   (a) Find the first five terms
   (b) Prove by induction that $u_n = \frac{9 \times 4^{n-1} + 11}{3}$
   (c) Calculate u_{18}
   (d) Find the sum of the first 15 terms
37. The sequence $\{y_n\}$ satisfies $y_n = 5y_{n-1} - 6y_{n-2}$ with $y_1 = 3$ and $y_2 = 9$.
   (a) Find the first six terms
   (b) Show that the characteristic equation is $r^2 - 5r + 6 = 0$
   (c) Solve to find $r = 3$ and $r = 2$
   (d) Use the general solution $y_n = A \cdot 3^n + B \cdot 2^n$ to find A and B
   (e) Write the explicit formula for y_n
38. Consider the series $\sum_{r=1}^{\infty} \frac{2}{r(r+4)}$.
   (a) Use partial fractions to show that $\frac{2}{r(r+4)} = \frac{1}{2}\left(\frac{1}{r} - \frac{1}{r+4}\right)$
   (b) Write out the first few terms and observe the telescoping pattern
   (c) Find the sum of the first n terms
   (d) Determine the sum to infinity
39. The Padovan sequence is defined by $P_1 = 1$, $P_2 = 1$, $P_3 = 1$, and $P_n = P_{n-2} + P_{n-3}$ for $n \geq 4$.
   (a) Write down the first 15 terms
   (b) Calculate the ratios $\frac{P_{n+1}}{P_n}$ for $n = 1, 2, 3, \dots, 14$
   (c) Show that these ratios approach the plastic number $\rho \approx 1.324$

(d) Investigate the characteristic equation $x^3 - x - 1 = 0$ and its real root

40. A clock pendulum loses energy on each swing. Each swing covers $\frac{9}{10}$ of the distance of the previous swing. The first swing covers 10 cm.

(a) Find the distance covered on the 15th swing
(b) Calculate the total distance traveled when the pendulum comes to rest
(c) Find the number of swings needed to cover 99% of the total distance
(d) If each swing takes time proportional to the square root of its distance, find the total time to rest

Section I: Applications and Problem Solving

41. A business loan of £75,000 is taken out at 7.5% annual compound interest. Monthly payments of £800 are made.

(a) Set up a recurrence relation for the amount owed after n months
(b) Find the amount owed after 30 months
(c) Determine how many months it takes to pay off the loan
(d) Calculate the total amount paid and the interest charged

42. A social media post goes viral. Each person who sees it shares it with 4 others every 2 hours. Initially, 20 people see the post.

(a) Model the number of people who see the post each period as a geometric sequence
(b) Find the number of new viewers after 24 hours
(c) After how many hours will the number of total viewers exceed 1 million?
(d) If the sharing rate decreases to 2.5 after 12 hours due to saturation, find the total viewers after 48 hours

43. A Sierpinski carpet fractal uses squares with areas forming the sequence: 81, 9, 1, $\frac{1}{9}$, $\frac{1}{81}$, ... cm².

(a) Find the total area of all the squares in the fractal
(b) If each square's perimeter is 4 times the square root of its area, find the total perimeter
(c) If cutting costs £1 per cm of perimeter, find the total cutting cost
(d) What percentage of the total area is contributed by the first 4 generations?

44. A chemical reactor contains 800 mg of a substance. Every minute, 12% decomposes, and 40 mg of fresh substance is added.

(a) Set up a recurrence relation for the amount after n minutes
(b) Find the amount present after 15 minutes
(c) Determine the long-term equilibrium amount
(d) After how many minutes is the amount within 3% of the equilibrium?

45. A pension scheme involves contributing £4000 in the first year, £4400 in the second year, £4840 in the third year, and so on (increasing by 10% each year) for 35 years.

(a) Model the annual contributions as a geometric sequence
(b) Find the total amount contributed over 35 years
(c) If each contribution earns 8% annual compound interest from when it's made, find the total value after 35 years
(d) Compare this with contributing £4000 annually at 8% compound interest for 35 years

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

For more resources and practice materials, visit: stepupmaths.co.uk