A Level Pure Mathematics
Practice Test 4: Integration

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Basic Integration - Polynomials

1. Find these indefinite integrals:
   (a) $\int (6x^2 - 5x + 4)\,dx$
   (b) $\int (4x^3 + x^2 - 3x - 2)\,dx$
   (c) $\int (5x^4 + 3x - 6)\,dx$
   (d) $\int (2x^2 - \frac{3}{4}x + 9)\,dx$
   (e) $\int (4x + 1)^2\,dx$
   (f) $\int (3x - 4)(2x + 1)\,dx$

2. Integrate these functions involving negative and fractional powers:
   (a) $\int x^{-5}\,dx$
   (b) $\int (5x^{-1} + 2x^{\frac{3}{4}})\,dx$
   (c) $\int \frac{4}{x^6}\,dx$
   (d) $\int \sqrt[5]{x}\,dx$
   (e) $\int \frac{5}{\sqrt{x}}\,dx$
   (f) $\int (3x^{\frac{7}{4}} - 4x^{-\frac{3}{4}})\,dx$

3. Find these integrals by expanding first:
   (a) $\int \frac{4x^3 - 2x^2 + 3x}{x}\,dx$
   (b) $\int \frac{x^2 - 25}{x}\,dx$
   (c) $\int \frac{(2x+3)^2}{x}\,dx$
   (d) $\int \frac{3x^3 + 8}{x^2}\,dx$

4. Evaluate these definite integrals:
   (a) $\int_2^5 (2x^2 + x - 1)\,dx$
   (b) $\int_1^4 (5x - 2)\,dx$
   (c) $\int_{-2}^1 x^3\,dx$
   (d) $\int_4^{16} \sqrt{x}\,dx$

5. Find the function $f(x)$ given:
   (a) $f'(x) = 8x^2 + 6x - 1$ and $f(0) = 6$
   (b) $f'(x) = 12x - 5$ and $f(1) = 10$
   (c) $f''(x) = 10x - 8$, $f'(0) = 4$, and $f(0) = 5$
   (d) $f'(x) = \frac{4}{x^5}$ for $x > 0$ and $f(1) = 2$

Section B: Integration of Standard Functions

6. Integrate these exponential and logarithmic functions:
   (a) $\int 7e^x \, dx$
   (b) $\int 8e^x \, dx$
   (c) $\int e^{5x} \, dx$
   (d) $\int e^{-4x} \, dx$
   (e) $\int \frac{5}{x} \, dx$ for $x > 0$
   (f) $\int \frac{7}{x} \, dx$

7. Integrate these trigonometric functions:
   (a) $\int 7 \sin x \, dx$
   (b) $\int 6 \cos x \, dx$
   (c) $\int 8 \sin x \, dx$
   (d) $\int 4 \cos x \, dx$
   (e) $\int 5 \sec^2 x \, dx$
   (f) $\int 4\text{cosec}^2 x \, dx$

8. Find these integrals:
   (a) $\int (4 \sin x - 3 \cos x) \, dx$
   (b) $\int (5e^x + 2x^3) \, dx$
   (c) $\int (4e^x + 3 \cos x) \, dx$
   (d) $\int \left(\frac{4}{x} - 3x\right) \, dx$ for $x > 0$
   (e) $\int (5 \sin x - 2e^{-x}) \, dx$
   (f) $\int \left(4x^2 - \frac{5}{x^2}\right) \, dx$ for $x > 0$

9. Evaluate these definite integrals:
   (a) $\int_0^{4\pi} \cos x \, dx$
   (b) $\int_0^{\frac{\pi}{4}} \sin x \, dx$
   (c) $\int_0^4 e^x \, dx$
   (d) $\int_1^{e^4} \frac{1}{x} \, dx$
   (e) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 x \, dx$
   (f) $\int_0^{\ln 5} e^{-x} \, dx$

10. Find the exact values:
   (a) $\int_0^{\frac{\pi}{2}} 5 \cos x \, dx$
   (b) $\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \sin x \, dx$
   (c) $\int_0^{\ln 5} 3e^x \, dx$
   (d) $\int_1^{e^3} \frac{5}{x} \, dx$

Section C: Integration by Substitution

11. Use substitution to find these integrals:
    (a) $\int (5x-3)^4 \, dx$
    (b) $\int (3x+4)^6 \, dx$
    (c) $\int x(4x^2-1)^3 \, dx$
    (d) $\int x\sqrt{3x^2-2} \, dx$
    (e) $\int \frac{4x}{3x^2+2} \, dx$
    (f) $\int \frac{5x}{(3x^2+1)^2} \, dx$
12. Find these integrals using appropriate substitutions:
    (a) $\int \sin(5x-3) \, dx$
    (b) $\int \cos(3x+\frac{\pi}{4}) \, dx$
    (c) $\int e^{5x-3} \, dx$
    (d) $\int e^{-5x} \, dx$
    (e) $\int \frac{1}{5x-3} \, dx$
    (f) $\int \frac{4}{3x+7} \, dx$
13. Use substitution for these more complex integrals:
    (a) $\int x^2(2x^3+5)^4 \, dx$
    (b) $\int \frac{x^2}{\sqrt{3x^3-1}} \, dx$
    (c) $\int xe^{4x^2} \, dx$
    (d) $\int \frac{\ln x}{4x} \, dx$
    (e) $\int \sin 4x \cos 3x \, dx$
    (f) $\int x \cot x \, dx$
14. Evaluate these definite integrals using substitution:
    (a) $\int_0^2 x(2x^2-1)^3 \, dx$
    (b) $\int_0^{\frac{\pi}{4}} \sin 4x \cos 2x \, dx$
    (c) $\int_1^3 \frac{3x}{2x^2-1} \, dx$
    (d) $\int_0^2 xe^{3x^2} \, dx$
15. Find these integrals by recognizing the derivative pattern:
    (a) $\int \frac{8x-3}{4x^2-3x+1} \, dx$
    (b) $\int \frac{9x^2-4}{3x^3-4x+2} \, dx$
    (c) $\int \frac{4e^x}{e^x+3} \, dx$
    (d) $\int \frac{3\cos x}{\sin x} \, dx$

Section D: Integration by Parts

16. Use integration by parts to find:
    (a) $\int 4xe^x\,dx$
    (b) $\int 3x\sin x\,dx$
    (c) $\int 3x\cos x\,dx$
    (d) $\int x^2e^{4x}\,dx$
    (e) $\int 4x\ln x\,dx$
    (f) $\int e^x\sin 3x\,dx$

17. Apply integration by parts to:
    (a) $\int 4\ln x\,dx$
    (b) $\int x^4\ln x\,dx$
    (c) $\int 3x\ln x\,dx$
    (d) $\int \ln(4x+1)\,dx$
    (e) $\int 2x\cos^{-1} x\,dx$
    (f) $\int x^2\sin 3x\,dx$

18. Find these integrals that may require multiple applications:
    (a) $\int x^2e^{-4x}\,dx$
    (b) $\int x^2\cos 3x\,dx$
    (c) $\int e^{4x}\cos 3x\,dx$
    (d) $\int e^{4x}\sin 3x\,dx$
    (e) $\int \cos(\ln 3x)\,dx$
    (f) $\int x^3e^{4x}\,dx$

19. Evaluate these definite integrals:
    (a) $\int_0^4 xe^x\,dx$
    (b) $\int_0^{\frac{\pi}{2}} x\cos x\,dx$
    (c) $\int_1^{e^4} x\ln x\,dx$
    (d) $\int_0^{\frac{\pi}{6}} x\sin 3x\,dx$

20. Prove these reduction formulas using integration by parts:
    (a) $I_n = \int x^ne^{4x}\,dx = \frac{x^ne^{4x}}{4} - \frac{n}{4}I_{n-1}$
    (b) $I_n = \int \sec^n x\,dx = \frac{\sec^{n-2} x\tan x}{n-1} + \frac{n-2}{n-1}I_{n-2}$ for $n \geq 2$
    (c) Use the first formula to find $\int x^4e^{4x}\,dx$

Section E: Area Under Curves

21. Find the area under these curves:
    (a) $y = 4x^2$ from $x = 0$ to $x = 3$
    (b) $y = 5x + 2$ from $x = 1$ to $x = 3$
    (c) $y = x^3 + 3x$ from $x = 0$ to $x = 2$
    (d) $y = 3\sin x$ from $x = 0$ to $x = \frac{\pi}{3}$

22. Calculate the area between the curve and the x-axis:
    (a) $y = x^2 - 16$ from $x = -4$ to $x = 4$
    (b) $y = x^3 - 9x$ from $x = -3$ to $x = 3$
    (c) $y = 3\cos x$ from $x = 0$ to $x = 2\pi$
    (d) $y = e^x - 4$ from $x = 0$ to $x = \ln 5$
23. Find the area between these curves:
    (a) $y = 4x^2$ and $y = 16$ from $x = 0$ to $x = 2$
    (b) $y = x^2$ and $y = 4x - 3$ from $x = 1$ to $x = 3$
    (c) $y = \sin 3x$ and $y = \cos 2x$ from $x = 0$ to $x = \frac{\pi}{6}$
    (d) $y = 3e^x$ and $y = 6$ from $x = 0$ to $x = \ln 2$
24. Find the total area enclosed by:
    (a) $y = x^2 - 16$ and the x-axis
    (b) $y = x^3 - 25x$ and the x-axis
    (c) $y = 3\sin x$ and $y = 0$ from $x = 0$ to $x = 2\pi$
    (d) $y = x^2 - 4x - 5$ and the x-axis
25. A region is bounded by $y = 4x^2$, $y = 0$, $x = 1$, and $x = 3$.
    (a) Calculate the area of the region
    (b) Find the x-coordinate of the centroid
    (c) Calculate the moment about the y-axis
    (d) Find the average value of $y = 4x^2$ over $[1, 3]$

Section F: Fundamental Theorem of Calculus

26. Use the fundamental theorem to evaluate:
    (a) $\frac{d}{dx}\int_0^x 4t^2\,dt$
    (b) $\frac{d}{dx}\int_4^x \cos t\,dt$
    (c) $\frac{d}{dx}\int_0^{4x} e^t\,dt$
    (d) $\frac{d}{dx}\int_{3x}^{x^2} \sin t\,dt$
27. Find these derivatives:
    (a) $\frac{d}{dx}\int_0^x \sqrt{16 + t^2}\,dt$
    (b) $\frac{d}{dx}\int_x^5 \frac{4}{t}\,dt$
    (c) $\frac{d}{dx}\int_{\sin 3x}^{\cos 2x} t^3\,dt$
    (d) $\frac{d}{dx}\int_0^{x^3} \cos(t^4)\,dt$
28. Given $K(x) = \int_3^x f(t)\,dt$ where f is continuous:
    (a) Prove that $K'(x) = f(x)$
    (b) If $f(x) = 4x^2 - 3$, find $K(x)$
    (c) Verify that $K'(x) = f(x)$ for your answer
    (d) Calculate $K(5) - K(4)$ and interpret geometrically

29. Solve these differential equations using antiderivatives:
    (a) $\frac{dy}{dx} = 8x^3 + 6x - 2$ with $y(0) = 5$
    (b) $\frac{dy}{dx} = 4e^x - \cos x$ with $y(0) = 3$
    (c) $\frac{d^2y}{dx^2} = 12x - 10$ with $y'(0) = 4$ and $y(0) = 3$
    (d) $\frac{dy}{dx} = \frac{4}{x}$ with $y(1) = 5$
30. For the function $k(x) = \int_4^x \frac{1}{t}\,dt$:
    (a) Find $k'(x)$
    (b) Show that $k(xy) = k(x) + k(y)$ for $x, y > 0$
    (c) Prove that $k(x^n) = n \cdot k(x)$ for $x > 0$ and integer n
    (d) Express $k(x)$ in terms of elementary functions

Section G: Volumes of Revolution

31. Find the volume when these curves are rotated about the x-axis:
    (a) $y = 4x$ from $x = 0$ to $x = 2$
    (b) $y = 3x^2$ from $x = 0$ to $x = 2$
    (c) $y = \sqrt{4x}$ from $x = 0$ to $x = 4$
    (d) $y = e^{4x}$ from $x = 0$ to $x = 1$
32. Calculate volumes of revolution about the x-axis:
    (a) $y = 3x + 1$ from $x = 0$ to $x = 2$
    (b) $y = x^2 - 3$ from $x = -2$ to $x = 2$
    (c) $y = 3\sin x$ from $x = 0$ to $x = \frac{\pi}{2}$
    (d) $y = \frac{4}{x}$ from $x = 1$ to $x = 4$
33. Find volumes when rotated about the y-axis:
    (a) $x = 4y^2$ from $y = 0$ to $y = 1$
    (b) $x = \sqrt{4y}$ from $y = 0$ to $y = 4$
    (c) $x = e^{4y}$ from $y = 0$ to $y = 1$
    (d) $x = 4\ln y$ from $y = 1$ to $y = e^4$
34. Use the washer method to find volumes:
    (a) Region between $y = 3x^2$ and $y = 12$ rotated about x-axis
    (b) Region between $y = 4x$ and $y = x^2$ rotated about x-axis
    (c) Region between $y = 3e^x$ and $y = 4$ from $x = 0$ to $x = \ln(\frac{4}{3})$ rotated about x-axis
    (d) Region between $y = \sqrt{4x}$ and $y = 3x$ rotated about y-axis
35. A solid has circular cross-sections. The radius at height h is $r(h) = \sqrt{25 - h^2}$ for $0 \leq h \leq 5$.
    (a) Set up the integral for the volume
    (b) Calculate the volume
    (c) Identify the shape of the solid
    (d) Find the surface area if this represents a hemisphere

Section H: Applications in Physics and Engineering

36. A particle moves with velocity $v(t) = 3t^2 - 8t + 4$ m/s.
    (a) Find the displacement from $t = 0$ to $t = 4$
    (b) Calculate the total distance traveled
    (c) Find the position function if $s(0) = 12$
    (d) Determine when the particle changes direction
    (e) Calculate the average velocity over $[0, 4]$

37. The acceleration of an object is $a(t) = 6t - 10$ m/s².
    (a) Find the velocity if $v(0) = 5$ m/s
    (b) Find the position if $s(0) = 3$
    (c) Calculate the displacement from $t = 2$ to $t = 4$
    (d) Find when the object is at rest
    (e) Determine the minimum position of the object

38. Water flows into a tank at rate $R(t) = 10 - 2t$ liters per minute.
    (a) Find the total volume added in the first 4 minutes
    (b) If the tank initially contains 30 liters, find $V(t)$
    (c) Calculate the average flow rate over 4 minutes
    (d) Find when the flow rate becomes zero
    (e) Determine the maximum volume in the tank

39. The magnetic field energy density is $u = \frac{B^2}{2\mu}$ where B is field strength.
    (a) Find total energy $U = \int u\,dV$ for uniform field in volume V
    (b) If $B(x) = B_0 \cos(\frac{\pi x}{L})$, find energy in region $0 \le x \le L$
    (c) Calculate energy if $B_0 = 0.5$ T, $L = 2$ m, and cross-sectional area is 0.1 m²
    (d) Compare with energy stored in inductor $U = \frac{1}{2}LI^2$

40. The power dissipated in a resistor follows $P(t) = I_0^2 R e^{-2t/\tau}$ where τ is time constant.
    (a) Find total energy $E = \int_0^\infty P(t)\,dt$ dissipated
    (b) Calculate energy dissipated in first time constant τ
    (c) Find the fraction of total energy dissipated by time $t = 2\tau$
    (d) Determine when half the total energy has been dissipated

Section I: Advanced Applications and Techniques

41. The center of mass of a thin rod from $x = a$ to $x = b$ with density $\rho(x)$ is: $\bar{x} = \frac{\int_a^b x\rho(x)\,dx}{\int_a^b \rho(x)\,dx}$
    (a) Find the center of mass of a rod from $x = 0$ to $x = 5$ with density $\rho(x) = 4x + 3$
    (b) Calculate the total mass of the rod
    (c) Find the center of mass if density is $\rho(x) = e^{4x}$
    (d) Compare with uniform density $\rho(x) = 4$

42. The moment of inertia about the x-axis is $I_x = \int y^2\,dm$ where $dm = \rho\,dA$.

(a) Find I_x for the region under $y = 4x^2$ from $x = 0$ to $x = 1$ with uniform density
(b) Calculate the radius of gyration $r_g = \sqrt{\frac{I_x}{M}}$
(c) Find the moment of inertia about the y-axis
(d) Explain the physical significance of the radius of gyration

43. Arc length of a curve $y = f(x)$ from $x = a$ to $x = b$ is: $L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$
    (a) Find the arc length of $y = 4x^2$ from $x = 0$ to $x = 1$
    (b) Calculate the arc length of $y = \ln(4x)$ from $x = 1$ to $x = e$
    (c) Find the perimeter of one arch of $y = 3 \sin x$
    (d) Derive the parametric arc length formula from first principles

44. Surface area of revolution about x-axis is: $S = 2\pi \int_a^b y\sqrt{1 + (y')^2}\, dx$
    (a) Find the surface area when $y = 4x$ from $x = 0$ to $x = 2$ is rotated
    (b) Calculate surface area for $y = \sqrt{4x}$ from $x = 0$ to $x = 4$
    (c) Find the surface area of a cone with base radius $3R$ and height $4h$
    (d) Verify using cone surface area formula $S = \pi r\sqrt{r^2 + h^2}$

45. Economic applications of integration:
    (a) If marginal cost is $MC(x) = 5x + 9$, find total cost function given fixed costs of £250
    (b) Calculate consumer surplus if demand is $p = 40 - 4x^2$ and price is £16
    (c) Find producer surplus for supply curve $p = 3x^2 + 5$ at equilibrium price £14
    (d) Calculate the effect on welfare of a price ceiling at £12

46. Probability density functions satisfy $\int_{-\infty}^{\infty} f(x)\, dx = 1$.
    (a) Find the constant e so that $f(x) = ex^5$ is a PDF on $[0, 1]$
    (b) Calculate $P(0.2 \leq X \leq 0.7)$ for this distribution
    (c) Find the mode (value where $f(x)$ is maximum)
    (d) Calculate the variance using $\text{Var}(X) = E[X^2] - (E[X])^2$

47. Design an integration problem modeling structural engineering:
    (a) Define a beam loading scenario with distributed forces
    (b) Set up integrals for shear force and bending moment
    (c) Solve for deflection using double integration
    (d) Interpret results for structural design criteria
    (e) Discuss safety factors and design limitations

48. Computational integration techniques:
    (a) Use the trapezoidal rule with $n = 10$ to approximate $\int_0^2 e^{-x^2}\, dx$
    (b) Apply Simpson's rule with $n = 10$ to the same integral
    (c) Compare with the error function $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$
    (d) Research applications of the error function in statistics
    (e) Discuss convergence properties of numerical methods

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

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