A Level Pure Mathematics
Practice Test 6: Proof

Instructions:
Answer all questions. Show your working clearly.
Calculators may NOT be used in this test.
Time allowed: 2 hours

Section A: Direct Proof

1. Prove that the product of two odd integers is always odd.
2. Prove that if n is an even integer, then n^2 is even.
3. Prove that the difference of the squares of two consecutive even integers is always divisible by 4.
4. Prove that for any integer n, the expression $n^2 + n$ is always even.
5. Given that a and b are irrational numbers such that $a+b$ is rational, prove that $a-b$ is irrational.
6. Prove that if $0 < x < y$, then $\frac{2xy}{x+y} < \sqrt{xy}$ (Harmonic-Geometric mean inequality).
7. Prove that for any real numbers a and b, $a^2 + b^2 \geq 2ab$ with equality if and only if $a = b$.
8. Prove that in any triangle with sides a, b, and c, we have $|a - b| < c < a + b$.
9. Let $f(x) = x^5 - 3x^3 + x$. Prove that f is an odd function.
10. Prove that the function $h(x) = -3x + 7$ is strictly decreasing on $\mathbb{R}$.

Section B: Proof by Contradiction

11. Prove that $\sqrt{5}$ is irrational.
12. Prove that there is no smallest positive rational number.
13. Prove that $\sqrt{6}$ is irrational.
14. Prove that if n^2 is odd, then n is odd.
15. Prove that there is no largest even integer.
16. Prove that if a and b are integers with $a^2 + b^2 = 5$, then at least one of a or b must be even.
17. Prove that $\log_3 2$ is irrational.
18. Prove that the equation $x^2 + 2 = 0$ has no real solutions.
19. Prove that the equation $x^2 - 5y^2 = 3$ has no integer solutions.
20. Prove that if p is prime and $p > 3$, then $p^2 - 1$ is divisible by 24.

Section C: Mathematical Induction - Sequences and Series

21. Prove by induction that $2 + 4 + 6 + \ldots + 2n = n(n+1)$ for all positive integers n.
22. Prove by induction that $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ for all positive integers n.
23. Prove by induction that $2 + 5 + 8 + \ldots + (3n-1) = \frac{n(3n+1)}{2}$ for all positive integers n.
24. Prove by induction that $1 + 5 + 9 + \ldots + (4n-3) = n(2n-1)$ for all positive integers n.
25. Prove by induction that $3 + 7 + 11 + \ldots + (4n-1) = n(2n+1)$ for all positive integers n.
26. Let $v_1 = 3$ and $v_{n+1} = 2v_n - 1$ for $n \geq 1$. Prove by induction that $v_n = 2^n + 1$ for all positive integers n.
27. Prove by induction that $\sum_{r=1}^{n} r \cdot 2^r = (n-1)2^{n+1} + 2$ for all positive integers n.
28. Prove by induction that $\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$ for all positive integers n.
29. For the sequence defined by $T_1 = 1$, $T_2 = 3$, and $T_{n+1} = T_n + 2T_{n-1}$ for $n \geq 2$, prove by induction that $T_n = 2^n - (-1)^n$ for all $n \geq 1$.
30. Prove by induction that $\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3}$ for all positive integers n.

Section D: Mathematical Induction - Inequalities

31. Prove by induction that $3^n \geq 2n + 1$ for all non-negative integers n.
32. Prove by induction that $2^n \geq n + 1$ for all non-negative integers n.
33. Prove by induction that $n! \geq 2^{n-1}$ for all positive integers n.
34. Prove by induction that $(1+x)^n \geq 1 + nx + \frac{n(n-1)}{2}x^2$ for all real $x \geq 0$ and all integers $n \geq 2$.
35. Prove by induction that $1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} < 2 - \frac{1}{n}$ for all integers $n \geq 2$.
36. Prove by induction that $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$ for all positive integers n.
37. Prove by induction that $1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} > \frac{3n}{2n+1}$ for all positive integers n.
38. Prove by induction that $3^n > 2n^2$ for all integers $n \geq 3$.
39. Prove by induction that $\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\ldots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$ for all integers $n \geq 2$.
40. Prove by induction that for $n \geq 2$, $\frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}} + \ldots + \frac{1}{\sqrt{2n}} > \frac{\sqrt{n}}{2}$.

Section E: Mathematical Induction - Divisibility

41. Prove by induction that $n^3 + 2n$ is divisible by 3 for all positive integers n.
42. Prove by induction that $5^n - 1$ is divisible by 4 for all positive integers n.
43. Prove by induction that $6^n - 1$ is divisible by 5 for all positive integers n.
44. Prove by induction that $n^3 - n$ is divisible by 6 for all positive integers n.
45. Prove by induction that $8^n - 1$ is divisible by 7 for all positive integers n.
46. Prove by induction that $3^{2n+1} + 2^{n+2}$ is divisible by 7 for all non-negative integers n.

47. Prove by induction that $9^n - 4^n$ is divisible by 5 for all positive integers n.
48. Prove by induction that $4^{2n} - 1$ is divisible by 15 for all positive integers n.
49. Prove by induction that $n^7 - n$ is divisible by 7 for all positive integers n.
50. Prove by induction that $10^n + 3 \cdot 4^{n+2} + 5$ is divisible by 9 for all non-negative integers n.

Section F: Deduction in Algebraic Manipulation

51. Given that $x + y = 7$ and $xy = 12$, find the value of $x^3 + y^3$.
52. If $a + b + c = 6$ and $ab + bc + ca = 11$, find the value of $a^2 + b^2 + c^2$.
53. Given that α and β are roots of $x^2 + px + q = 0$, prove that:
    (a) $\alpha + \beta = -p$
    (b) $\alpha\beta = q$
    (c) $\alpha^3 + \beta^3 = -p^3 + 3pq$
54. If $x - \frac{1}{x} = 3$, find expressions for:
    (a) $x^2 + \frac{1}{x^2}$
    (b) $x^3 - \frac{1}{x^3}$
    (c) $x^4 + \frac{1}{x^4}$
55. Prove that if $a + b + c = 0$, then $(a + b + c)^3 = 3(a + b)(b + c)(c + a)$.
56. Given that p, q, r are in geometric progression, prove that $q^2 = pr$.
57. If $\sin A + \sin B + \sin C = 0$ and $\cos A + \cos B + \cos C = 0$, prove that $\sin 3A + \sin 3B + \sin 3C = 3\sin(A + B + C)$.
58. Prove that $(a - b)^3 + (b - c)^3 + (c - a)^3 = 3(a - b)(b - c)(c - a)$.
59. Given that x, y, z are in arithmetic progression, prove that $x^2 + z^2 = 2(y^2 + xz - xy - yz)$.
60. If a, b, c are the roots of $x^3 - px^2 + qx - r = 0$, prove that $a + b + c = p$.

Section G: Deduction in Geometric Reasoning

61. In triangle ABC, prove that the exterior angle is equal to the sum of the two non-adjacent interior angles.
62. Prove that if two chords of a circle intersect, then the product of their segments are equal.
63. Prove that the angle subtended by an arc at the center is twice the angle subtended by the same arc at any point on the circle.
64. In triangle ABC, let M be the midpoint of side BC. Prove that $AM^2 = \frac{2AB^2+2AC^2-BC^2}{4}$.
65. Prove that the diagonals of a rhombus are perpendicular bisectors of each other.
66. In a circle, prove that the perpendicular from the center to a chord passes through the midpoint of the arc subtended by the chord.
67. Prove that the angle between two tangents drawn from an external point equals half the difference of the intercepted arcs.
68. In triangle ABC, prove that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ (sine rule).
69. Prove that in any triangle, the three angle bisectors meet at a single point (the incenter).
70. Prove that the three altitudes of a triangle meet at a single point (the orthocenter).

Section H: Advanced Proof Techniques

71. Prove that between any two distinct irrational numbers, there exists a rational number.

72. Prove that if $f(x) = \frac{ax+b}{cx+d}$ where $ad - bc \neq 0$ and $cx + d \neq 0$, then f is bijective on its domain.

73. Prove that the set of rational numbers is countably infinite.

74. Use the pigeonhole principle to prove that in any group of 6 people, either 3 are mutual friends or 3 are mutual strangers.

75. Prove that $\sqrt{2} + \sqrt{5}$ is irrational.

76. Prove that if $\gcd(a, n) = 1$ and $\gcd(b, n) = 1$, then $\gcd(ab, n) = 1$.

77. Prove that the product of a non-zero rational number and an irrational number is irrational.

78. Use strong induction to prove that every positive integer can be written as a sum of distinct powers of 2.

79. Prove that if $a_1, a_2, \ldots, a_n$ are positive real numbers, then:

$$\sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}}$$

(GM-HM inequality)

80. Prove or disprove: The number $2^n + 1$ is prime for all non-negative integers n.

Section I: Proof Writing and Communication

81. Write a complete proof that for any triangle with sides a, b, c and semiperimeter s, the area is $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ (Heron's formula).

82. Prove that the equation $x^4 + y^4 = 2z^4$ has no positive integer solutions.

83. Let $P_n = 1 \cdot 3 \cdot 5 \cdots (2n-1)$ be the product of the first n odd numbers. Prove that $P_n = \frac{(2n)!}{2^n \cdot n!}$.

84. Prove Cauchy's inequality: For real numbers a_1, a_2, b_1, b_2:

$$(a_1 b_1 + a_2 b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2)$$

85. Consider the sequence defined by $c_1 = 2$, $c_2 = 3$, and $c_{n+2} = c_{n+1} + c_n$ for $n \geq 1$. Prove that c_n and c_{n+1} are coprime for all $n \geq 1$.

86. Prove that for any positive integer n, the number $3^{2n+1} + 2^{n+2}$ is divisible by 7.

87. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = \frac{x+1}{x-1}$ for $x \neq 1$. Prove that g is bijective on its domain and find its inverse function.

88. Prove Fermat's Little Theorem: If p is prime and a is not divisible by p, then $a^{p-1} \equiv 1 \pmod{p}$.

89. Prove that π is irrational. (You may use the fact that if π were rational, then $\sin(\pi) = 0$ would lead to a contradiction with known properties of the sine function)

90. Write a proof showing that there exist rational numbers r and s such that r^s is irrational, but s^r is rational.

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

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