A Level Pure Mathematics
Practice Test 6: Exponentials and Logarithms

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Laws of Indices and Exponential Functions

1. Simplify these expressions using laws of indices:
   (a) $11^4 \times 11^3 \times 11^{-5}$
   (b) $\frac{12^8 \times 12^{-3}}{12^4}$
   (c) $(2^5)^2 \times 2^{-6}$
   (d) $\frac{(10^3)^2 \times 10^{-4}}{10^3}$
   (e) $(6^4 \times 6^{-3})^2$
   (f) $\frac{32^{3x} \times 32^{x+3}}{32^{5x-2}}$

2. Solve these exponential equations:
   (a) $11^x = 1331$
   (b) $12^{x-1} = 1728$
   (c) $2^{4x+1} = 128$
   (d) $64^x = \frac{1}{8}$
   (e) $121^x = 11^{x+6}$
   (f) $25^{2x} = 125^{x+1}$

3. Express in the form a^x where a is a rational number:
   (a) $(\frac{1}{9})^x \times 81^x$
   (b) $\frac{243^x}{9^{3x}}$
   (c) $(343)^{\frac{x}{3}} \times (\frac{1}{7})^x$
   (d) $\frac{100^x \times 10^{-2x}}{1000^{\frac{x}{3}}}$

4. Sketch the graphs of these exponential functions, showing key features:
   (a) $y = 11^x$
   (b) $y = (\frac{1}{11})^x$
   (c) $y = 9^x - 4$
   (d) $y = 8^{x+2}$

(e) $y = -8^x$
(f) $y = 8^{-x}$

5. For the function $f(x) = 6e^x$:

(a) State the domain and range
(b) Find the y-intercept
(c) Describe the behavior as $x \to \infty$ and $x \to -\infty$
(d) Find $f'(x)$ and comment on the gradient
(e) Sketch the graph, showing the tangent at $(0, 6)$

Section B: Logarithmic Functions and Properties

6. Express these in logarithmic form:

(a) $11^3 = 1331$
(b) $10^{-6} = 0.000001$
(c) $e^p = 8$
(d) $12^0 = 1$
(e) $6^{-3} = \frac{1}{216}$
(f) $j^k = l$

7. Express these in exponential form:

(a) $\log_{11} 1331 = 3$
(b) $\log_{10} 0.000001 = -6$
(c) $\ln e^8 = 8$
(d) $\log_6 \frac{1}{216} = -3$
(e) $\log_j l = k$
(f) $\ln e^{10} = 10$

8. Evaluate these logarithms without a calculator:

(a) $\log_{11} 121$
(b) $\log_{12} 1728$
(c) $\log_{10} 100000000$
(d) $\log_2 \frac{1}{128}$
(e) $\log_{81} 27$
(f) $\log_{64} 8$

9. Use the laws of logarithms to simplify:

(a) $\log_j 9 + \log_j 11$
(b) $\log_j 72 - \log_j 9$
(c) $8 \log_j 5$
(d) $\log_j x + 7 \log_j y - \log_j z$
(e) $\frac{1}{7} \log_j 128 + \log_j 6$
(f) $\log_j(v^2 - 64) - \log_j(v - 8)$ where $v > 8$

10. Sketch the graphs of these logarithmic functions:

(a) $y = \log_{11} x$
(b) $y = \ln x$
(c) $y = \log_{11} x - 5$
(d) $y = \log_{11}(x + 5)$
(e) $y = -\log_{11} x$
(f) $y = \log_{11}(-x)$ for $x < 0$

Section C: Solving Logarithmic Equations

11. Solve these logarithmic equations:

    (a) $\log_{11} x = 2$
    (b) $\log_{12}(x - 5) = 2$
    (c) $\log_{10}(7x + 3) = 2$
    (d) $\ln(x + 7) = 0$
    (e) $\log_3(x^2) = 8$
    (f) $7\log_4 x = 14$

12. Solve these equations involving multiple logarithms:

    (a) $\log_{11} x + \log_{11} 3 = 2$
    (b) $\log_j 30 - \log_j 10 = \log_j x$
    (c) $\log_6 x + \log_6(x - 7) = 2$
    (d) $\log_{10} x - \log_{10}(x - 8) = \log_{10} 7$
    (e) $7\log_5 x = \log_5 25$
    (f) $\log_4(x + 6) + \log_4(x - 6) = 3$

13. Solve these equations where the base is unknown:

    (a) $\log_a 81 = 2$
    (b) $\log_a \frac{1}{343} = -3$
    (c) $\log_a 4096 = \frac{12}{3}$
    (d) $\log_a 16807 = \frac{10}{2}$

14. Solve these quadratic logarithmic equations:

    (a) $(\log_{11} x)^2 = 4$
    (b) $(\log_6 x)^2 - 8\log_6 x + 12 = 0$
    (c) $\log^2 x - 8\log x + 12 = 0$ (base 10)
    (d) $(\ln x)^2 - 10\ln x + 21 = 0$

15. Use the change of base formula to evaluate:

    (a) $\log_{11} 17$ in terms of natural logarithms
    (b) $\log_{12} 32$ in terms of common logarithms
    (c) $\log_9 25$ using ln
    (d) Express $\log_p q \times \log_q r \times \log_r p$

Section D: Combined Exponential and Logarithmic Equations

16. Solve these mixed equations:
    (a) $e^x = 17$
    (b) $11^x = 55$
    (c) $12 \times 5^x = 300$
    (d) $8^{x+3} = 256$
    (e) $e^{2x} - 8e^x + 12 = 0$
    (f) $11^{2x} - 12 \times 11^x + 11 = 0$

17. Solve using substitution methods:
    (a) $121^x - 11^{x+3} - 1331 = 0$ (let $y = 11^x$)
    (b) $128^x - 15 \times 2^x + 2 = 0$ (let $u = 2^x$)
    (c) $e^{2x} - 11e^x + 28 = 0$ (let $t = e^x$)
    (d) $\log^2 x - 8 \log x + 12 = 0$ (let $z = \log x$)

18. Find the exact solutions:
    (a) $\ln x + \ln(x - 7) = \ln 18$
    (b) $\log_5 x + \log_{125} x = 3$
    (c) $e^x + e^{-x} = 8$
    (d) $7 \ln x = \ln(x + 30)$

19. Solve these equations involving both exponentials and logarithms:
    (a) $x = \log_{11}(11^x + 10)$
    (b) $e^{\ln x} = x + 11$
    (c) $\ln(e^x - 6) = 6$
    (d) $\log_7(7^x + 42) = x + 4$

20. Find the values of x for which:
    (a) $11^x > 14641$
    (b) $\log_{12} x < 2$
    (c) $e^x \leq 35$
    (d) $\ln x \geq 5$
    (e) $\log_{11}(x - 6) > 1$
    (f) $12^{x-3} < \frac{1}{144}$

Section E: Exponential Growth and Decay

21. A population grows according to $P = P_0 e^{kt}$ where $P_0 = 2400$ and $k = 0.08$ per year.
    (a) Find the population after 2 years
    (b) How long for the population to increase by 100%?
    (c) What is the percentage growth rate per year?
    (d) Find when the population reaches 7200
    (e) Calculate the population after 9 years

22. A radioactive substance decays according to $A = A_0 e^{-\lambda t}$ where $\lambda = 0.0866$ per year.
    (a) If initially there are 400g, find the amount after 4 years
    (b) Calculate the half-life of the substance
    (c) How long for 90% to decay?
    (d) What percentage remains after 15 years?
    (e) Find when only 40g remains

23. An investment grows at 12% compound interest per annum.
    (a) Write the growth formula
    (b) How long to increase by 300%?
    (c) If £1200 is invested, find the value after 18 years
    (d) How long for the investment to reach £6000?
    (e) Compare with simple interest at 12% over 4 years

24. The temperature of a cooling object follows Newton's law: $T = T_{\text{room}} + (T_0 - T_{\text{room}})e^{-kt}$
    (a) If room temperature is 26°C, initial temperature is 98°C, and $k = 0.04$ per minute, find the temperature after 25 minutes
    (b) How long for the object to cool to 52°C?
    (c) Find the temperature after 75 minutes
    (d) What happens as $t \to \infty$?
    (e) If the object cools to 80°C after 12 minutes, find k

25. Carbon-14 dating uses the formula $A = A_0 e^{-0.000121t}$ where t is in years.
    (a) Calculate the half-life of carbon-14
    (b) If a sample has 50% of its original carbon-14, find its age
    (c) How old is a sample with 2% remaining?
    (d) What percentage remains after 35000 years?
    (e) Find the age of a sample with ratio 0.35 of living organisms

Section F: Logarithmic Modeling and Applications

26. The Richter scale for earthquake magnitude is given by $M = \log_{10}(\frac{I}{I_0})$ where I is intensity.
    (a) If one earthquake has magnitude 8.5 and another has magnitude 6.5, compare their intensities
    (b) An earthquake has intensity 500000 times the reference level. Find its magnitude
    (c) How much more intense is a magnitude 10 earthquake than magnitude 8?
    (d) Find the magnitude of an earthquake with intensity $6 \times 10^5 I_0$

27. The pH scale is defined as $\text{pH} = -\log_{10}[\text{H}^+]$ where $[\text{H}^+]$ is hydrogen ion concentration.
    (a) Find the pH when $[\text{H}^+] = 10^{-8}$ mol/L
    (b) If pH = 1.2, find the hydrogen ion concentration
    (c) Compare the acidity of solutions with pH 0 and pH 10
    (d) Find the pH when $[\text{H}^+] = 8.7 \times 10^{-2}$ mol/L
    (e) If the concentration is multiplied by 10, how does the pH change?

28. Sound intensity level in decibels is $L = 10\log_{10}(\frac{I}{I_0})$ where $I_0 = 10^{-12}$ W/m**2**.

    (a) Find the decibel level when $I = 10^{-1}$ W/m**2**
    (b) A sound has level 115 dB. Find its intensity
    (c) How much more intense is 130 dB than 90 dB?
    (d) Find the intensity of a 15 dB sound
    (e) If intensity increases by factor 10000, by how much do decibels increase?

29. The Michaelis-Menten equation in biochemistry is $v = \frac{V_{\max}[S]}{K_m+[S]}$.

    (a) Take logarithms to linearize when $[S] >> K_m$
    (b) If $V_{\max} = 200$, $K_m = 20$, find v when $[S] = 40$
    (c) Plot $\log v$ against $\log[S]$ for large $[S]$
    (d) Find $[S]$ when $v = \frac{4V_{\max}}{5}$

30. In information theory, entropy is $H = -\sum p_i \log_2 p_i$.

    (a) For a fair 64-sided die, calculate the entropy
    (b) For a biased coin with $P(\text{H}) = 0.85$, find the entropy
    (c) Find the entropy of a fair 128-sided die
    (d) What probability distribution maximizes entropy for 8 outcomes?

Section G: Advanced Functions and Transformations

31. Analyze the function $f(x) = \ln(x + 7) - 6$:

    (a) State the domain and range
    (b) Find the x and y intercepts
    (c) Identify any asymptotes
    (d) Find $f^{-1}(x)$
    (e) Sketch both $f(x)$ and $f^{-1}(x)$

32. For the function $g(x) = e^{7x-2} - 8$:

    (a) Describe the transformations from $y = e^x$
    (b) State the domain and range
    (c) Find the horizontal asymptote
    (d) Solve $g(x) = 0$
    (e) Find $g^{-1}(x)$

33. Consider $h(x) = \log_7(49 - x^2)$:

    (a) Find the domain of $h(x)$
    (b) Determine the range
    (c) Find the maximum value and where it occurs
    (d) Solve $h(x) = 1$
    (e) Sketch the graph of $y = h(x)$

34. The function $k(x) = te^{sx} + r$ passes through $(0, 15)$, $(1, 24)$, and has horizontal asymptote $y = 7$.

    (a) Find the values of t, s, and r

(b) Write the equation of $k(x)$
(c) Find $k(2)$
(d) Solve $k(x) = 35$
(e) Find the domain and range of $k(x)$

35. Investigate the function $m(x) = x^4 \ln x$ for $x > 0$:
(a) Find $m'(x)$ and $m''(x)$
(b) Locate any stationary points
(c) Determine the nature of stationary points
(d) Find the behavior as $x \to 0^+$ and $x \to \infty$
(e) Sketch the graph of $y = m(x)$

Section H: Systems and Complex Problems

36. Solve these simultaneous equations:
(a) $\begin{cases} y = 7^x \\ y = 14 - x \end{cases}$
(b) $\begin{cases} \ln y = 7x \\ y = e^{x+6} \end{cases}$
(c) $\begin{cases} \log_7 x + \log_7 y = 2 \\ x + y = 28 \end{cases}$
(d) $\begin{cases} e^x + e^y = 16 \\ e^x - e^y = 12 \end{cases}$

37. Find where these curves intersect:
(a) $y = e^x$ and $y = \ln x$
(b) $y = 7^x$ and $y = x^7$
(c) $y = \log x$ and $y = 7 - x$
(d) $y = e^{-x}$ and $y = x + 6$

38. Solve these differential equations:
(a) $\frac{dy}{dx} = ky$ where $y(0) = y_0$
(b) $\frac{dP}{dt} = rP(1 - \frac{P}{K})$ (logistic growth)
(c) $\frac{dT}{dt} = -k(T - T_{\text{env}})$ (Newton's cooling)
(d) $\frac{dN}{dt} = -\lambda N$ (radioactive decay)

39. A bacteria culture follows logistic growth: $P(t) = \frac{L}{1+ae^{-kt}}$
(a) If $L = 3000$, $P(0) = 120$, and $P(1) = 240$, find a and k
(b) Find the population after 10 days
(c) When does the population reach 1500?
(d) Find the maximum growth rate and when it occurs
(e) Compare with exponential growth $P = 120e^{rt}$

40. The Arrhenius equation in chemistry is $k = Ae^{-E_a/(RT)}$ where k is reaction rate.

(a) Take natural logarithms to linearize the equation
(b) If at temperature 330K, $k = 0.018$, and at 390K, $k = 0.20$, find E_a/R
(c) Find the activation energy if $R = 8.314$ J/(mol·K)
(d) Predict the rate constant at 440K
(e) At what temperature does the rate increase 8-fold from 330K?

Section I: Advanced Applications and Modeling

41. A pharmacokinetic model describes drug concentration: $C(t) = \frac{D}{V}e^{-kt}$ where D is dose, V is volume of distribution, k is elimination rate.

    (a) If $D = 1200$ mg, $V = 80$ L, $k = 0.20$ h^{-1}, find the initial concentration
    (b) Calculate the concentration after 4 hours
    (c) Find the half-life of the drug
    (d) When does concentration drop to 4 mg/L?
    (e) Model repeated dosing every 2 hours

42. Economic growth follows $Y(t) = Y_0e^{rt}$ where r is the growth rate.

    (a) If GDP grows at 10% per year, how long to double?
    (b) A country's GDP is £6 trillion and grows to £9.6 trillion in 5 years. Find the growth rate
    (c) Compare linear growth $Y = Y_0(1 + rt)$ with exponential over 45 years
    (d) Find when exponential growth overtakes linear with same initial rate
    (e) Model with varying growth rate $r(t) = r_0e^{-\zeta t}$

43. The spread of an epidemic follows $I(t) = \frac{N}{1+(N/I_0-1)e^{-rt}}$ (logistic model).

    (a) If $N = 30000$, $I_0 = 35$, $r = 0.08$ per day, find infections after 20 days
    (b) When do infections peak?
    (c) Find the maximum rate of spread
    (d) Compare with exponential model $I = I_0e^{rt}$ for early stages
    (e) Model intervention reducing r by 90% after day 26

44. Weber-Fechner law relates stimulus and perception: $P = k\log(S/S_0)$.

    (a) If increasing stimulus 7-fold increases perception by 35 units, find k
    (b) Find perception when stimulus increases 25-fold
    (c) A sound's loudness follows $L = 10\log_{10}(I/I_0)$. Compare two sounds differing by 45 dB
    (d) Model brightness perception where threshold $S_0 = 0.20$ lux
    (e) Explain why Weber-Fechner law applies to many sensory modalities

45. Design an optimization problem involving exponentials:

    (a) A company's profit is $P(t) = 3500e^{0.01t} - 1200t$ over t years
    (b) Find when profit is maximized
    (c) Calculate maximum profit
    (d) Determine break-even points
    (e) Model with discounting: present value $= \frac{P(t)}{e^{rt}}$
    (f) Find optimal time to sell considering 10% discount rate

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

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