A Level Pure Mathematics
Practice Test 1: Proof

Instructions:
Answer all questions. Show your working clearly.
Calculators may NOT be used in this test.
Time allowed: 2 hours

Section A: Direct Proof

1. Prove that the sum of two even integers is always even.
2. Prove that if n is an odd integer, then n^2 is odd.
3. Prove that the sum of the squares of two consecutive integers is always odd.
4. Prove that for any integer n, the expression $n(n+1)$ is always even.
5. Given that a and b are rational numbers, prove that $a+b$ is rational.
6. Prove that if $x > 0$ and $y > 0$, then $\frac{x+y}{2} \geq \sqrt{xy}$ (AM-GM inequality).
7. Prove that for any real numbers a and b, $(a+b)^2 \geq 4ab$ if and only if $a \geq 0$ and $b \geq 0$.
8. Prove that if a, b, and c are the sides of a triangle, then $a+b > c$, $a+c > b$, and $b+c > a$.
9. Let $f(x) = x^3 + x$. Prove that f is an odd function.
10. Prove that the function $g(x) = 2x + 3$ is strictly increasing on $\mathbb{R}$.

Section B: Proof by Contradiction

11. Prove that $\sqrt{2}$ is irrational.
12. Prove that there are infinitely many prime numbers.
13. Prove that $\sqrt{3}$ is irrational.
14. Prove that if n^2 is even, then n is even.
15. Prove that there is no largest rational number.
16. Prove that if a and b are integers with $a^2 + b^2 = 3$, then at least one of a or b is zero.
17. Prove that $\log_2 3$ is irrational.
18. Prove that if x is real and $x^2 + 1 = 0$, then x is not real. (Show this leads to a contradiction)
19. Prove that the equation $x^2 - 3y^2 = 2$ has no integer solutions.
20. Prove that if p is prime and $p > 2$, then p is odd.

Section C: Mathematical Induction - Sequences and Series

21. Prove by induction that $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ for all positive integers n.
22. Prove by induction that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all positive integers n.
23. Prove by induction that $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ for all positive integers n.
24. Prove by induction that $2 + 4 + 6 + \ldots + 2n = n(n + 1)$ for all positive integers n.
25. Prove by induction that $1 + 4 + 7 + \ldots + (3n - 2) = \frac{n(3n-1)}{2}$ for all positive integers n.
26. Let $u_1 = 2$ and $u_{n+1} = 3u_n + 1$ for $n \geq 1$. Prove by induction that $u_n = \frac{3^n - 1}{2}$ for all positive integers n.
27. Prove by induction that $\sum_{r=1}^{n} r \cdot r! = (n + 1)! - 1$ for all positive integers n.
28. Prove by induction that $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$ for all positive integers n.
29. The Fibonacci sequence is defined by $F_1 = 1$, $F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$. Prove by induction that $F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$ for all $n \geq 1$.
30. Prove by induction that $\sum_{r=1}^{n} r^3 = \left(\frac{n(n+1)}{2}\right)^2$ for all positive integers n.

Section D: Mathematical Induction - Inequalities

31. Prove by induction that $2^n > n$ for all positive integers n.
32. Prove by induction that $3^n \geq 2n + 1$ for all non-negative integers n.
33. Prove by induction that $n! > 2^n$ for all integers $n \geq 4$.
34. Prove by induction that $(1+x)^n \geq 1+nx$ for all real $x \geq 0$ and all positive integers n (Bernoulli's inequality).
35. Prove by induction that $\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{n^2} < 2 - \frac{1}{n}$ for all integers $n \geq 2$.
36. Prove by induction that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} > 2(\sqrt{n+1} - 1)$ for all positive integers n.
37. Prove by induction that $1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} < 2 - \frac{1}{n}$ for all integers $n \geq 2$.
38. Prove by induction that $2^n > n^2$ for all integers $n \geq 5$.
39. Prove by induction that $\left(1 + \frac{1}{n}\right)^n < 3$ for all positive integers n.
40. Prove by induction that for $n \geq 1$, $\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} > \frac{13}{24}$.

Section E: Mathematical Induction - Divisibility

41. Prove by induction that $n^3 - n$ is divisible by 3 for all positive integers n.
42. Prove by induction that $4^n - 1$ is divisible by 3 for all positive integers n.
43. Prove by induction that $5^n - 1$ is divisible by 4 for all positive integers n.
44. Prove by induction that $n^3 + 2n$ is divisible by 3 for all positive integers n.
45. Prove by induction that $7^n - 1$ is divisible by 6 for all positive integers n.
46. Prove by induction that $3^{2n} - 1$ is divisible by 8 for all positive integers n.

47. Prove by induction that $11^n - 6^n$ is divisible by 5 for all positive integers n.

48. Prove by induction that $2^{3n} - 1$ is divisible by 7 for all positive integers n.

49. Prove by induction that $n^5 - n$ is divisible by 5 for all positive integers n.

50. Prove by induction that $13^n - 1$ is divisible by 12 for all positive integers n.

Section F: Deduction in Algebraic Manipulation

51. Given that $x + y = 5$ and $xy = 6$, find the value of $x^2 + y^2$.

52. If $a + b + c = 0$, prove that $a^3 + b^3 + c^3 = 3abc$.

53. Given that α and β are roots of $x^2 - px + q = 0$, prove that:
    (a) $\alpha + \beta = p$
    (b) $\alpha\beta = q$
    (c) $\alpha^2 + \beta^2 = p^2 - 2q$

54. If $x + \frac{1}{x} = k$, find expressions for:
    (a) $x^2 + \frac{1}{x^2}$
    (b) $x^3 + \frac{1}{x^3}$
    (c) $x^4 + \frac{1}{x^4}$

55. Prove that if $a + b + c = 0$, then $a^2 + b^2 + c^2 = -2(ab + bc + ca)$.

56. Given that p, q, r are in arithmetic progression, prove that $(p - r)^2 = 4(p - q)(q - r)$.

57. If $\tan A + \tan B + \tan C = \tan A \tan B \tan C$, prove that $A + B + C = \pi$.

58. Prove that $(a + b + c)^3 = a^3 + b^3 + c^3 + 3(a + b)(b + c)(c + a)$.

59. Given that x, y, z are in geometric progression, prove that $\log x$, $\log y$, $\log z$ are in arithmetic progression.

60. If a, b, c are in harmonic progression, prove that $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ are in arithmetic progression.

Section G: Deduction in Geometric Reasoning

61. In triangle ABC, prove that the sum of any two sides is greater than the third side.

62. Prove that the perpendicular from the center of a circle to a chord bisects the chord.

63. Prove that the angle in a semicircle is a right angle.

64. In triangle ABC, let D, E, F be the midpoints of sides BC, CA, AB respectively. Prove that triangle DEF is similar to triangle ABC with ratio $1 : 2$.

65. Prove that the diagonals of a parallelogram bisect each other.

66. In a circle, prove that equal chords subtend equal angles at the center.

67. Prove that tangents from an external point to a circle are equal in length.

68. In triangle ABC, prove that $a^2 = b^2 + c^2 - 2bc\cos A$ (cosine rule).

69. Prove that in any triangle, the three medians meet at a single point (the centroid).

70. Prove that the perpendicular bisectors of the sides of a triangle meet at a single point (the circumcenter).

Section H: Advanced Proof Techniques

71. Prove that between any two distinct rational numbers, there exists another rational number.

72. Prove that if $f(x) = ax + b$ where $a \neq 0$, then f is bijective from $\mathbb{R}$ to $\mathbb{R}$.

73. Prove that the set of even integers has the same cardinality as the set of all integers.

74. Use the pigeonhole principle to prove that among any 13 people, at least two share the same birth month.

75. Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

76. Prove that if p is prime and p divides ab, then p divides a or p divides b.

77. Prove that the sum of a rational number and an irrational number is irrational.

78. Use strong induction to prove that every integer greater than 1 can be expressed as a product of prime numbers.

79. Prove that if $a_1, a_2, \ldots, a_n$ are positive real numbers, then:

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

(AM-GM inequality for n terms)

80. Prove or disprove: For all positive integers n, $n^2 + n + 41$ is prime.

Section I: Proof Writing and Communication

81. Write a complete proof that for any triangle with sides a, b, c and area Δ, the radius of the inscribed circle is $r = \frac{\Delta}{s}$ where $s = \frac{a+b+c}{2}$.

82. Prove that the equation $x^4 + 4y^4 = z^2$ has no positive integer solutions. (Hint: Consider the equation modulo appropriate values)

83. Let $S_n = 1^k + 2^k + 3^k + \ldots + n^k$ for some fixed positive integer k. Prove that S_n is a polynomial in n of degree $k + 1$.

84. Prove Lagrange's identity: For real numbers a_1, a_2, b_1, b_2:

$$(a_1^2 + a_2^2)(b_1^2 + b_2^2) = (a_1 b_1 + a_2 b_2)^2 + (a_1 b_2 - a_2 b_1)^2$$

85. Consider the sequence defined by $a_1 = 1$, $a_2 = 1$, and $a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$. Prove that $\gcd(a_n, a_{n+1}) = 1$ for all $n \geq 1$.

86. Prove that for any positive integer n, the number $4^n + 6n - 1$ is divisible by 9.

87. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3$. Prove that f is bijective and find its inverse function.

88. Prove Wilson's theorem: If p is prime, then $(p - 1)! \equiv -1 \pmod{p}$.

89. Prove that e (Euler's number) is irrational. (You may use the series expansion $e = \sum_{n=0}^{\infty} \frac{1}{n!}$)

90. Write a proof by contradiction showing that there exist irrational numbers a and b such that a^b is rational.

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

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