A Level Pure Mathematics
Practice Test 3: Numerical Methods

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Introduction to Numerical Methods

1. Explain why numerical methods are needed for the following equations:
   (a) $x^3 + 5x - 8 = 0$
   (b) $x = \cot x$
   (c) $e^x = 5x$
   (d) $x^5 - 4x^3 + 2x - 4 = 0$

2. For each function, determine the approximate location of roots by examining sign changes:
   (a) $f(x) = x^3 - 5x - 4$ for $x \in [-3, 4]$
   (b) $f(x) = x^2 - 6x - 2$ for $x \in [-2, 7]$
   (c) $f(x) = e^x - 4x - 1$ for $x \in [-1, 3]$
   (d) $f(x) = \ln x - x + 4$ for $x \in [3, 6]$

3. State the conditions required for the Intermediate Value Theorem and explain how it guarantees the existence of roots.

4. For the function $f(x) = x^3 - 3x - 3$:
   (a) Show that there is a root between $x = 2$ and $x = 3$
   (b) Determine a more precise interval containing the root
   (c) Sketch the graph of $y = f(x)$ showing the root location
   (d) Explain why this equation cannot be solved algebraically

5. Define the following terms in the context of numerical methods:
   (a) Absolute error
   (b) Relative error
   (c) Tolerance
   (d) Convergence
   (e) Iteration
   (f) Fixed point

Section B: Bisection Method

6. Use the bisection method to find the root of $f(x) = x^3 - 5x - 8$ in the interval $[2, 3]$.
   (a) Complete 4 iterations
   (b) Give your answer correct to 2 decimal places
   (c) Estimate the error in your final approximation
   (d) How many iterations would be needed for an accuracy of 10^{-7}?

7. Apply the bisection method to solve $x = \cot x$:
   (a) Show that a root lies between 0.1 and 1
   (b) Perform 5 iterations starting with $[0.1, 1]$
   (c) Give your answer to 4 decimal places
   (d) Verify your answer by substitution

8. Use the bisection method to find the positive root of $x^2 - 7x - 3 = 0$:
   (a) Determine a suitable starting interval
   (b) Perform iterations until the root is accurate to 3 decimal places
   (c) Compare with the exact solution using the quadratic formula
   (d) Calculate the absolute error

9. For the equation $e^x = 5x$:
   (a) Show graphically that there are two roots
   (b) Use bisection to find the smaller positive root to 3 decimal places
   (c) Find the larger root to 3 decimal places
   (d) Discuss the convergence rate of the bisection method

10. The equation $\ln x = 5 - x$ has a root near $x = 4$.
    (a) Use bisection method with initial interval $[3, 5]$
    (b) Continue until consecutive approximations differ by less than 0.01
    (c) How many iterations were required?
    (d) What is the theoretical minimum number of iterations needed?

Section C: Newton-Raphson Method

11. Use the Newton-Raphson method to solve $x^3 - 5x - 8 = 0$ starting with $x_0 = 2.8$.
    (a) Write down the iteration formula
    (b) Perform 4 iterations
    (c) Give your answer to 5 decimal places
    (d) Compare the convergence with bisection method

12. Apply Newton-Raphson to find $\sqrt{15}$ by solving $x^2 - 15 = 0$:
    (a) Derive the iteration formula
    (b) Start with $x_0 = 4$ and perform 4 iterations
    (c) Compare with the exact value
    (d) Explain why this converges so quickly

13. Solve $\cot x = x$ using Newton-Raphson method:
    (a) Rearrange to standard form $f(x) = 0$
    (b) Find $f'(x)$ and write the iteration formula
    (c) Use $x_0 = 0.6$ and perform 5 iterations
    (d) Check your answer by substitution

14. Use Newton-Raphson to solve $e^x - 4x - 1 = 0$:
    (a) Find the iteration formula
    (b) Starting with $x_0 = 2$, find the root to 6 decimal places
    (c) Starting with $x_0 = 0$, find the other root
    (d) Discuss the importance of choosing good initial values

15. Investigate the convergence of Newton-Raphson for $f(x) = x^3 - 9x + 6$:
    (a) Find all roots using different starting values
    (b) Identify cases where the method fails to converge
    (c) Explain why some starting values lead to divergence
    (d) Sketch the function and its derivative to illustrate your findings

16. For the equation $x^4 - 10x^2 + 21 = 0$:
    (a) Solve exactly by substitution
    (b) Use Newton-Raphson to find all four roots
    (c) Compare numerical and exact solutions
    (d) Discuss which starting values work best for each root

Section D: Fixed Point Iteration

17. Rearrange $x^2 - 6x + 4 = 0$ into the form $x = g(x)$ in different ways:
    (a) $x = \frac{x^2+4}{6}$
    (b) $x = 6 - \frac{4}{x}$
    (c) $x = \sqrt{6x - 4}$
    (d) Test each rearrangement for convergence near $x = 5.236$

18. Use fixed point iteration to solve $x = \cot x$:
    (a) Use the iteration $x_{n+1} = \cot x_n$ with $x_0 = 0.6$
    (b) Perform 10 iterations
    (c) Plot the values to show convergence
    (d) Explain why this method converges

19. Solve $x^3 - 3x - 3 = 0$ using fixed point iteration:
    (a) Try the rearrangement $x = x^3 - 3$
    (b) Try the rearrangement $x = \sqrt[3]{3x + 3}$
    (c) Try the rearrangement $x = \frac{3}{x^2-3}$
    (d) Determine which rearrangements converge and why

20. For the equation $e^x = 5x$:

(a) Show that $x = \frac{e^x}{5}$ diverges from $x_0 = 1$
(b) Try $x = \ln(5x)$ starting from $x_0 = 1.2$
(c) Explain the convergence behavior using $|g'(x)|$
(d) Find the root to 4 decimal places

21. Investigate the convergence condition $|g'(x)| < 1$:

(a) For $g(x) = \frac{x^2+4}{6}$, find $g'(x)$ and determine convergence regions
(b) For $g(x) = \sqrt{6x-4}$, analyze convergence
(c) Explain why some iterations converge while others diverge
(d) Relate this to the graphical interpretation of fixed point iteration

Section E: Trapezium Rule

22. Use the trapezium rule with 4 strips to approximate:

(a) $\int_0^2 x^4\,dx$
(b) $\int_1^3 \frac{1}{x^3}\,dx$
(c) $\int_0^1 e^{3x}\,dx$
(d) $\int_0^{\pi/2} \sin 2x\,dx$

Compare with exact values and calculate absolute errors.

23. Apply the trapezium rule to $\int_0^1 \sqrt{1+x^4}\,dx$:

(a) Use 2 strips, then 4 strips, then 8 strips
(b) Comment on how the approximation improves
(c) Estimate the true value of the integral
(d) Explain why exact integration is difficult

24. For $\int_1^2 \frac{1}{x^4}\,dx$:

(a) Calculate the exact value
(b) Use trapezium rule with $n = 2, 4, 8$ strips
(c) Calculate the error for each approximation
(d) Show that halving the strip width approximately quarters the error

25. Use the trapezium rule to estimate $\int_0^1 e^{-x^4}\,dx$:

(a) Use 5 ordinates (4 strips)
(b) Use 9 ordinates (8 strips)
(c) Compare your answers and estimate the accuracy
(d) This integral cannot be expressed in elementary functions - explain why numerical methods are essential

26. A curve passes through points $(0, 2.5)$, $(0.5, 3.2)$, $(1, 4.1)$, $(1.5, 3.7)$, $(2, 2.8)$:

(a) Use trapezium rule to find the area under the curve
(b) If the y-values represent velocity in m/s and x represents time in seconds, interpret your answer
(c) How could you improve the accuracy?
(d) Discuss the limitations when working with discrete data points

Section F: Simpson's Rule

27. Use Simpson's rule with 4 strips to approximate:

    (a) $\int_0^2 x^5 \, dx$
    (b) $\int_1^3 \frac{1}{x^4} \, dx$
    (c) $\int_0^1 e^{4x} \, dx$
    (d) $\int_0^\pi \sin 2x \, dx$

    Compare with exact values and trapezium rule approximations.

28. Apply Simpson's rule to $\int_0^1 \frac{1}{1+x^4} \, dx$:

    (a) Use 2 strips, then 4 strips, then 8 strips
    (b) Comment on the convergence pattern
    (c) Compare convergence with trapezium rule
    (d) Explain why Simpson's rule is more accurate

29. For $\int_0^4 \sqrt{16 - x^2} \, dx$:

    (a) Recognize this as the area of a quarter circle
    (b) Use Simpson's rule with 4 and 8 strips
    (c) Compare with the exact value 4π
    (d) Calculate percentage errors

30. Use Simpson's rule to estimate $\int_1^3 x^3 \ln x \, dx$:

    (a) Use 4 strips
    (b) Use 8 strips
    (c) The exact value is $\frac{81 \ln 3 - 20}{4}$ - verify this and calculate errors
    (d) Discuss the convergence rate

31. A bridge has the following cross-sectional areas at 6m intervals:

Distance (m)	0	6	12	18	24	30
Area (m^2)	75	105	130	125	95	70

    (a) Use Simpson's rule to estimate the volume of material
    (b) If concrete costs £80 per m^3, estimate the total material cost
    (c) Discuss the accuracy of your approximation
    (d) What additional data would improve the estimate?

Section G: Error Analysis and Comparison of Methods

32. For $\int_0^1 x^7 \, dx$:

    (a) Calculate the exact value
    (b) Use trapezium rule with $n = 2, 4, 8$ strips
    (c) Use Simpson's rule with $n = 2, 4, 8$ strips
    (d) Create a table comparing errors
    (e) Verify the theoretical error formulas

33. Analyze the errors in numerical integration:
    (a) Explain why trapezium rule has error proportional to h^2
    (b) Explain why Simpson's rule has error proportional to h^4
    (c) For what types of functions is each method most accurate?
    (d) Give examples where each method might be preferred

34. Compare root-finding methods for $f(x) = x^3 - 5x - 8$:
    (a) Use bisection method (6 iterations from $[2, 3]$)
    (b) Use Newton-Raphson (4 iterations from $x_0 = 2.8$)
    (c) Use fixed point iteration with $x = \sqrt[3]{5x + 8}$ (6 iterations from $x_0 = 2.8$)
    (d) Compare convergence rates and accuracy
    (e) Discuss advantages and disadvantages of each method

35. For the equation $\csc x = x$ in $(0, \pi)$:
    (a) Explain why bisection method works reliably
    (b) Discuss potential problems with Newton-Raphson method
    (c) Suggest appropriate starting values and intervals
    (d) Find the root using your preferred method

36. Error propagation in numerical methods:
    (a) If $f(2.8) = 0.18$ and $f(3) = -0.32$, estimate the error in the root found by linear interpolation
    (b) For Newton-Raphson, if $f'(x)$ is small near the root, how does this affect convergence?
    (c) In numerical integration, how do rounding errors accumulate?
    (d) Suggest strategies to minimize computational errors

Section H: Advanced Applications

37. Solve systems of equations numerically. For the system: $x^2 + y^2 = 13$, $xy = 4$
    (a) Rearrange to eliminate one variable
    (b) Solve the resulting equation using Newton-Raphson
    (c) Find all solutions
    (d) Verify your answers by substitution
    (e) Compare with algebraic solution

38. A projectile's height is given by $h(t) = 35t - 5t^2$ for $t \geq 0$.
    (a) Find when the projectile hits the ground exactly
    (b) Use numerical methods to find when $h(t) = 25$
    (c) Find the maximum height and when it occurs
    (d) Use numerical integration to find the total distance traveled
    (e) Model air resistance with $h(t) = 35t - 5t^2 - 0.12t^3$ and solve numerically

39. The equation $x^3 - 6x + d = 0$ has parameter d.
    (a) For what values of d does the equation have three real roots?
    (b) For $d = 4$, find all roots numerically

(c) For $d = -4$, find all roots numerically
(d) Investigate the behavior as d varies
(e) Create a bifurcation diagram showing how roots change with d

40. Population growth is modeled by $\frac{dP}{dt} = rP(1 - \frac{P}{K})$ where $P(0) = P_0$.
    (a) For $r = 0.08$, $K = 1500$, $P_0 = 100$, the solution is $P(t) = \frac{1500}{1+14e^{-0.08t}}$
    (b) Use Newton-Raphson to find when $P(t) = 750$
    (c) Find when the growth rate $\frac{dP}{dt}$ is maximum
    (d) Use numerical integration to find the total growth in the first 30 years
    (e) Model seasonal variation with $r(t) = 0.08 + 0.02\sin(2\pi t)$ and solve numerically

41. Financial modeling: An investment grows according to $A(t) = Pe^{rt}$ where r varies.
    (a) If $P = 2500$ and $A(7) = 3800$, find r using Newton-Raphson
    (b) For compound interest $A = P(1 + \frac{r}{n})^{nt}$, find r when $P = 2500$, $A = 5000$, $t = 15$, $n = 2$
    (c) Use numerical integration to find the average value of $A(t)$ over $[0, 15]$
    (d) Model variable interest rates and compare investment strategies

Section I: Advanced Topics and Optimization

42. Multi-variable Newton-Raphson for system: $f(x, y) = x^2 + y^2 - 7 = 0$, $g(x, y) = xy - 3 = 0$
    (a) Set up the Jacobian matrix
    (b) Derive the iteration formulas
    (c) Find a solution starting from $(2.5, 1.5)$
    (d) Compare with single-variable approach
    (e) Discuss convergence criteria for systems

43. Optimization using numerical methods:
    (a) Find the minimum of $f(x) = x^4 - 8x^3 + 18x^2 - 12x + 5$ using Newton-Raphson on $f'(x) = 0$
    (b) Use numerical integration to find the area under $f(x)$ from 0 to 5
    (c) Find the point where $f(x) = 3$ has multiple solutions
    (d) Analyze the stability of each critical point

44. Adaptive integration methods:
    (a) Implement Richardson extrapolation for trapezium rule
    (b) Use adaptive Simpson's rule with error control
    (c) Compare computational efficiency
    (d) Apply to $\int_0^1 \frac{\cot x}{x} dx$ (using $\lim_{x\to 0} \frac{\cot x}{x} = \frac{1}{x}$)

45. Boundary value problems: Solve $y'' + 4y = x$ with $y(0) = 0$, $y(\pi) = 0$:
    (a) Convert to a system of first-order equations
    (b) Use shooting method with Newton-Raphson
    (c) Implement finite difference method
    (d) Compare solutions with exact answer
    (e) Discuss numerical stability

46. Fourier analysis using numerical methods:
    (a) For $f(x) = x$ on $[-\pi, \pi]$, compute Fourier coefficients numerically
    (b) Use trapezium rule to evaluate $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx$
    (c) Compute b_n coefficients similarly
    (d) Compare with analytical Fourier series
    (e) Discuss convergence and Gibbs phenomenon
47. Chaos and sensitivity analysis:
    (a) Study the logistic map $x_{n+1} = rx_n(1 - x_n)$
    (b) For $r = 3.4$, find the fixed point using Newton-Raphson
    (c) For $r = 3.6$, demonstrate chaotic behavior
    (d) Show sensitivity to initial conditions
    (e) Create a bifurcation diagram for $r \in [3.0, 4]$
    (f) Discuss implications for numerical accuracy
48. Monte Carlo methods:
    (a) Estimate π using random points in a unit square
    (b) Use Monte Carlo integration for $\int_0^1 e^{-x^5}dx$
    (c) Compare accuracy with deterministic methods
    (d) Analyze convergence rate ($\propto 1/\sqrt{n}$)
    (e) Discuss when Monte Carlo methods are preferred
    (f) Apply to multi-dimensional integration
49. Numerical differentiation and applications:
    (a) Derive forward, backward, and central difference formulas
    (b) Estimate $f'(2.5)$ for $f(x) = \sin(x^2)$ using different step sizes
    (c) Analyze truncation error vs. rounding error trade-off
    (d) Apply to find critical points of tabulated data
    (e) Use for solving differential equations numerically
    (f) Implement higher-order difference formulas
50. Spline interpolation and curve fitting:
    (a) Construct cubic spline through points $(0, 2.5)$, $(1, 3.1)$, $(2, 2.8)$, $(3, 3.4)$
    (b) Compare with polynomial interpolation
    (c) Discuss advantages of splines for numerical integration
    (d) Apply to data smoothing problems
    (e) Use for solving differential equations
    (f) Implement natural and clamped boundary conditions
51. Design a comprehensive numerical analysis project:
    (a) Choose a real-world problem requiring multiple numerical methods
    (b) Implement root-finding, integration, and optimization
    (c) Analyze error propagation and computational complexity
    (d) Compare different numerical approaches
    (e) Validate results against known solutions where possible
    (f) Present findings with appropriate visualizations
    (g) Discuss limitations and potential improvements

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 200

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