A Level Pure Mathematics
Practice Test 5: Sequences and Series

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Arithmetic Sequences

1. For the arithmetic sequence $13, 21, 29, 37, 45, \ldots$:
   - (a) Find the first term a and common difference d
   - (b) Find the general term u_n
   - (c) Calculate u_{26}
   - (d) Find which term equals 189
   - (e) Determine if 250 is a term in the sequence
2. An arithmetic sequence has $u_7 = 41$ and $u_{13} = 71$.
   - (a) Find the first term and common difference
   - (b) Write the general term u_n
   - (c) Calculate u_{25}
   - (d) Find the first term to exceed 120
   - (e) Determine the largest value of n for which $u_n < 200$
3. The nth term of an arithmetic sequence is $u_n = 7n - 4$.
   - (a) Write down the first five terms
   - (b) Find the common difference
   - (c) Calculate u_{38}
   - (d) Find the sum of the first 28 terms
   - (e) For what value of n is $u_n = 199$?
4. Three numbers $r - 5h$, r, and $r + 5h$ are in arithmetic progression with sum 72 and product 13824.
   - (a) Find the value of r
   - (b) Set up an equation for h
   - (c) Solve to find the values of h
   - (d) Write down the three numbers for each case
5. An arithmetic sequence has first term a and common difference d.

(a) If the sum of the first m terms is k times the sum of the first n terms, prove that $\frac{m}{n} = \frac{k[2a+(n-1)d]}{2a+(m-1)d}$

(b) Show that if the pth, qth, and rth terms are in geometric progression, then $(q-r)a = (r-p-q+r)d$

(c) Prove that the sum of n terms can be written as $S_n = \frac{n}{2}(first\ term + last\ term)$

(d) If $S_p = S_q$ where $p \neq q$, show that the $(p+q)$th term is zero when $p+q$ is odd

Section B: Arithmetic Series

6. Calculate the sum of these arithmetic series:

   (a) $10 + 17 + 24 + 31 + \ldots$ (first 24 terms)

   (b) $40 + 35 + 30 + 25 + \ldots$ (first 16 terms)

   (c) $\frac{3}{7} + \frac{5}{7} + 1 + \frac{9}{7} + \ldots$ (first 28 terms)

   (d) The series with first term 21, last term 147, and 19 terms

7. An arithmetic series has first term 13 and common difference 8.

   (a) Find the sum of the first 22 terms

   (b) Find the smallest value of n for which $S_n \geq 3000$

   (c) If the sum of the first n terms is 2288, find n

   (d) Express S_n in terms of n

8. The sum of the first n terms of an arithmetic series is $S_n = 5n^2 + 3n$.

   (a) Find the first term u_1

   (b) Find u_2 and u_3

   (c) Determine the common difference

   (d) Find the general term u_n

   (e) Verify using the formula $u_n = S_n - S_{n-1}$ for $n \geq 2$

9. Find the sum of:

   (a) All multiples of 11 between 100 and 1000

   (b) All integers from 1 to 150 that are divisible by 8

   (c) All even integers from 20 to 180

   (d) The integers from 1 to 120 that are divisible by 3 or 7

10. An arithmetic series has $S_{16} = 552$ and $S_{24} = 1128$.

   (a) Find the first term and common difference

   (b) Calculate S_{35}

   (c) Find the 20th term

   (d) Determine when the sum first exceeds 4000

Section C: Geometric Sequences

11. For the geometric sequence $6, 24, 96, 384, 1536, \ldots$:
    (a) Find the first term a and common ratio r
    (b) Find the general term u_n
    (c) Calculate u_{13}
    (d) Find which term equals 6144
    (e) Determine if 98304 is a term in the sequence
12. A geometric sequence has $u_6 = 96$ and $u_9 = 768$.
    (a) Find the common ratio r
    (b) Find the first term a
    (c) Write the general term u_n
    (d) Calculate u_{14}
    (e) Find the first term to exceed 500000
13. The nth term of a geometric sequence is $u_n = 9 \times 3^{n-1}$.
    (a) Write down the first five terms
    (b) Find the common ratio
    (c) Calculate u_{12}
    (d) Find the sum of the first 8 terms
    (e) For what value of n is $u_n = 6561$?
14. Three numbers $\frac{s}{t}$, s, and st are in geometric progression with sum 156 and product 2197.
    (a) Find the value of s
    (b) Set up an equation for t
    (c) Solve to find the values of t
    (d) Write down the three numbers for each case
15. A geometric sequence has first term a and common ratio r.
    (a) If the product of the first n terms is P_n, prove that $P_n = a^n \cdot r^{0+1+2+\ldots+(n-1)} = a^n \cdot r^{\frac{n(n-1)}{2}}$
    (b) Show that if u_p, u_q, u_r are in arithmetic progression, then $\frac{1}{u_p}$, $\frac{1}{u_q}$, $\frac{1}{u_r}$ are in harmonic progression
    (c) Prove that the sequence $\log u_1$, $\log u_2$, $\log u_3$, ... forms an arithmetic sequence
    (d) If two geometric sequences have the same common ratio, show that their term-by-term products also form a geometric sequence

Section D: Geometric Series

16. Calculate the sum of these geometric series:
    (a) $11 + 33 + 99 + 297 + \ldots$ (first 12 terms)
    (b) $4 - 16 + 64 - 256 + \ldots$ (first 9 terms)
    (c) $\frac{3}{7} + \frac{3}{21} + \frac{3}{63} + \frac{3}{189} + \ldots$ (first 10 terms)
    (d) $128 + 96 + 72 + 54 + \ldots$ (first 16 terms)
17. A geometric series has first term 18 and common ratio $\frac{4}{9}$.

(a) Find the sum of the first 25 terms
(b) Find the smallest value of n for which $S_n \geq 32$
(c) Calculate the sum to infinity
(d) Find how many terms are needed for the sum to be within 0.001 of the sum to infinity

18. The sum of the first n terms of a geometric series is $S_n = 7(6^n - 1)$.
   (a) Find the first term u_1
   (b) Find u_2 and u_3
   (c) Determine the common ratio
   (d) Find the general term u_n
   (e) Verify using the formula $u_n = S_n - S_{n-1}$ for $n \geq 2$

19. Evaluate these infinite geometric series:
   (a) $1 + \frac{3}{7} + \frac{9}{49} + \frac{27}{343} + \dots$
   (b) $12 - 6 + 3 - \frac{3}{2} + \dots$
   (c) $\frac{9}{10} + \frac{9}{50} + \frac{9}{250} + \frac{9}{1250} + \dots$
   (d) $0.5 + 0.05 + 0.005 + 0.0005 + \dots$

20. A geometric series has $S_7 = 254$ and $S_{14} = 16766$.
   (a) Set up equations for the first term and common ratio
   (b) Solve to find a and r
   (c) Calculate S_{21}
   (d) Find the sum to infinity (if it exists)
   (e) Determine the first term to exceed 50000

Section E: Sigma Notation

21. Evaluate these sums:
   (a) $\sum_{r=1}^{20}(6r + 4)$
   (b) $\sum_{r=1}^{40}(7r - 6)$
   (c) $\sum_{r=1}^{32} r^2$
   (d) $\sum_{r=1}^{18}(5r^2 + 2r)$

22. Express these series using sigma notation:
   (a) $12 + 19 + 26 + 33 + \dots + 68$
   (b) $8 + 40 + 200 + 1000 + \dots + 25000$
   (c) $3^3 + 5^3 + 7^3 + 9^3 + \dots + 21^3$
   (d) $\frac{1}{6} + \frac{1}{16} + \frac{1}{30} + \frac{1}{48} + \dots + \frac{1}{120}$

23. Use the standard formulae to evaluate:
   (a) $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$: Find $\sum_{r=1}^{95} r$
   (b) $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$: Find $\sum_{r=1}^{40} r^2$
   (c) $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$: Find $\sum_{r=1}^{25} r^3$
   (d) $\sum_{r=1}^{50}(6r^2 - 5r + 4)$

24. Simplify these expressions:
    (a) $\sum_{r=1}^{n}(tr+v)$ in terms of t, v, and n
    (b) $\sum_{r=1}^{n}(5r^2-3r+1)$
    (c) $\sum_{r=1}^{n}(4r+2)^2$
    (d) $\sum_{r=1}^{n} r(5r+3)$

25. Prove these results:
    (a) $\sum_{r=1}^{n}(6r-5)=n(3n-2)$
    (b) $\sum_{r=1}^{n} r(r+5)=\frac{n(n+1)(n+14)}{3}$
    (c) $\sum_{r=1}^{n} \frac{1}{(5r-4)(5r+1)}=\frac{n}{5n+1}$
    (d) $\sum_{r=1}^{n}((r+3)^2-(r+2)^2)=n(n+7)$

Section F: Binomial Expansion - Integer Powers

26. Expand using the binomial theorem:
    (a) $(x+6)^6$
    (b) $(6x-5)^4$
    (c) $(5-4x)^5$
    (d) $(5x+\frac{4}{x})^6$

27. Find the specified terms in these expansions:
    (a) The coefficient of x^7 in $(6x+4)^{11}$
    (b) The coefficient of x^9 in $(4x-3)^{12}$
    (c) The constant term in $(x^6+\frac{5}{x^4})^5$
    (d) The coefficient of x^{-3} in $(6x^4-\frac{3}{x^2})^8$

28. Use the binomial theorem to evaluate:
    (a) $(1.06)^7$ to 6 decimal places
    (b) $(0.93)^6$ to 5 decimal places
    (c) $(1.03)^9$ exactly
    (d) 102^5 by writing it as $(100+2)^5$

29. In the expansion of $(1+ex)^f$:
    (a) The coefficient of x is 24 and the coefficient of x^2 is 252. Find e and f.
    (b) Find the coefficient of x^3
    (c) Write out the first four terms of the expansion
    (d) For what values of x does the expansion converge?

30. The coefficient of x^k in the expansion of $(1+x)^f$ is $\binom{f}{k}$.
    (a) Show that $\sum_{k=0}^{f}\binom{f}{k}\cdot 4^k=5^f$
    (b) Prove that $\binom{f}{0}-\binom{f}{2}+\binom{f}{4}-\binom{f}{6}+\ldots=2^{f-1}$ when f is even
    (c) Use Vandermonde's identity to show that $\binom{m}{0}\binom{n}{k}+\binom{m}{1}\binom{n}{k-1}+\ldots+\binom{m}{k}\binom{n}{0}=\binom{m+n}{k}$
    (d) Show that $\sum_{k=0}^{f}\binom{f}{k}\cdot k\cdot 2^k=f\cdot 3^{f-1}$ for $f\geq 1$

Section G: Binomial Expansion - Non-Integer Powers

31. Expand these expressions up to and including the term in x^3:
    (a) $(1+x)^{4/5}$
    (b) $(1-x)^{-5}$
    (c) $(1+7x)^{2/3}$
    (d) $(1-8x)^{-2/3}$

32. Find the first four terms in the expansion of:
    (a) $(49+x)^{1/2}$
    (b) $(36-x)^{-1/2}$
    (c) $\frac{1}{(5+x)^5}$
    (d) $\sqrt{25-6x}$

33. State the range of values of x for which these expansions are valid:
    (a) $(1+7x)^{-1} = 1 - 7x + 49x^2 - 343x^3 + \ldots$
    (b) $(1-6x)^{1/2} = 1 - 3x - \frac{9x^2}{2} - 9x^3 - \ldots$
    (c) $(8+x)^{-1} = \frac{1}{8} - \frac{x}{64} + \frac{x^2}{512} - \frac{x^3}{4096} + \ldots$
    (d) $\frac{1}{\sqrt{36-x}} = \frac{1}{6} + \frac{x}{432} + \frac{x^2}{15552} + \ldots$

34. Use binomial expansions to find approximations:
    (a) $\sqrt{1.10}$ to 5 decimal places
    (b) $\frac{1}{\sqrt{0.84}}$ to 4 decimal places
    (c) $(1.06)^{-6}$ to 6 decimal places
    (d) $\sqrt[3]{1.15}$ to 5 decimal places

35. Find the coefficient of x^2 in the expansion of:
    (a) $(1+x)^{4/5}(1-x)^{1/5}$
    (b) $(1+6x)^{-1}(1+4x)^2$
    (c) $\frac{1+4x}{\sqrt{1-x}}$
    (d) $(1+3x-2x^2)(1+x)^{-5}$

Section H: Mixed Series and Advanced Topics

36. A sequence is defined by $u_1 = 6$ and $u_{n+1} = 5u_n - 13$ for $n \geq 1$.
    (a) Find the first five terms
    (b) Prove by induction that $u_n = \frac{11\times5^{n-1}+13}{4}$
    (c) Calculate u_{20}
    (d) Find the sum of the first 18 terms

37. The sequence $\{x_n\}$ satisfies $x_n = 6x_{n-1} - 8x_{n-2}$ with $x_1 = 4$ and $x_2 = 12$.
    (a) Find the first six terms
    (b) Show that the characteristic equation is $r^2 - 6r + 8 = 0$
    (c) Solve to find $r = 4$ and $r = 2$

(d) Use the general solution $x_n = A \cdot 4^n + B \cdot 2^n$ to find A and B

(e) Write the explicit formula for x_n

38. Consider the series $\sum_{r=1}^{\infty} \frac{3}{r(r+5)}$.

    (a) Use partial fractions to show that $\frac{3}{r(r+5)} = \frac{3}{5}\left(\frac{1}{r} - \frac{1}{r+5}\right)$
    (b) Write out the first few terms and observe the telescoping pattern
    (c) Find the sum of the first n terms
    (d) Determine the sum to infinity

39. The Narayana sequence is defined by $N_1 = 1$, $N_2 = 1$, and $N_n = N_{n-1} + N_{n-3}$ for $n \geq 4$ with $N_3 = 1$.

    (a) Write down the first 16 terms
    (b) Calculate the ratios $\frac{N_{n+1}}{N_n}$ for the first several terms
    (c) Investigate the periodic behavior of these ratios
    (d) Study the characteristic equation $x^3 - x^2 - 1 = 0$ and its roots

40. A ball bounces with decreasing height. Each bounce reaches $\frac{5}{6}$ of the height of the previous bounce. The initial drop is from 12 meters.

    (a) Find the height reached after the 20th bounce
    (b) Calculate the total vertical distance traveled when the ball comes to rest
    (c) Find the number of bounces needed for the height to fall below 1 meter
    (d) If each bounce takes time proportional to the height, find the ratio of total time to initial drop time

Section I: Applications and Problem Solving

41. A student loan of £45,000 is taken out at 6.5% annual compound interest. Annual payments of £6000 are made.

    (a) Set up a recurrence relation for the amount owed after n years
    (b) Find the amount owed after 8 years
    (c) Determine how many years it takes to pay off the loan
    (d) Calculate the total interest paid over the life of the loan

42. A rumor spreads through a school. Each person tells the rumor to 5 others every day. Initially, 3 people know the rumor.

    (a) Model the number of people who know the rumor each day as a geometric sequence
    (b) Find the number of people who know the rumor after 6 days
    (c) After how many days will more than 10,000 people know the rumor?
    (d) If the school has 2000 students and the spread rate decreases after 70% know, model this change

43. A Koch snowflake fractal uses triangular segments with perimeters forming the sequence: 108, 36, 12, 4, $\frac{4}{3}$, ... cm.

    (a) Find the total perimeter of all the triangular segments
    (b) If each segment's area is proportional to the square of its perimeter with constant $\frac{1}{48}$, find the total area

(c) If drawing costs £0.50 per cm of perimeter, find the total drawing cost
(d) What percentage of the total perimeter is contributed by the first 3 iterations?

44. A cooling system maintains temperature by removing heat. Initially, the system contains 600 units of heat. Every hour, 15% is removed, and 50 units are added from external sources.

(a) Set up a recurrence relation for the heat units after n hours
(b) Find the heat units present after 20 hours
(c) Determine the long-term equilibrium heat level
(d) After how many hours is the heat level within 2% of the equilibrium?

45. A savings plan involves depositing £5000 in the first year, £5500 in the second year, £6050 in the third year, and so on (increasing by 10% each year) for 30 years.

(a) Model the annual deposits as a geometric sequence
(b) Find the total amount deposited over 30 years
(c) If each deposit earns 7.5% annual compound interest from when it's made, find the total value after 30 years
(d) Compare this with depositing £5000 annually at 7.5% compound interest for 30 years

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

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