A Level Pure Mathematics
Practice Test 4: Proof

Instructions:
Answer all questions. Show your working clearly.
Calculators may NOT be used in this test.
Time allowed: 2 hours

Section A: Direct Proof

1. Prove that the product of an even integer and any integer is always even.
2. Prove that if n is an odd integer, then $n^2 - 1$ is even.
3. Prove that the square of any odd integer is of the form $8k + 1$ for some integer k.
4. Prove that for any integer n, the expression $n^2 - n$ is always even.
5. Given that a and b are rational numbers with $b \neq 0$, prove that $\frac{a}{b}$ is rational.
6. Prove that if $x \geq 0$ and $y \geq 0$, then $\sqrt{xy} \leq \frac{x+y}{2}$ with equality if and only if $x = y$.
7. Prove that for any real numbers m and n, $(m + n)^2 \geq 4mn$ if and only if $m \geq 0$ and $n \geq 0$.
8. Prove that in any triangle, the longest side is opposite to the largest angle.
9. Let $p(x) = x^9 - 4x^7 + 2x^5 - x^3 + 3x$. Prove that p is an odd function.
10. Prove that the function $q(x) = 3x + 7$ is strictly increasing on $\mathbb{R}$.

Section B: Proof by Contradiction

11. Prove that $\sqrt{13}$ is irrational.
12. Prove that between any two consecutive integers, there is no integer.
13. Prove that $\sqrt{15}$ is irrational.
14. Prove that if n^2 is divisible by 5, then n is divisible by 5.
15. Prove that there is no rational number r such that $r^2 = 3$.
16. Prove that if a and b are integers with $a^2 + 3b^2 = 4$, then $b = 0$.
17. Prove that $\log_2 5$ is irrational.
18. Prove that the equation $2x^2 + 3x + 4 = 0$ has no real solutions.
19. Prove that the equation $3x^2 - 2y^2 = 5$ has no integer solutions.
20. Prove that if p is a prime number and $p \geq 5$, then $p^2 - 1$ is divisible by 24.

Section C: Mathematical Induction - Sequences and Series

21. Prove by induction that $6 + 12 + 18 + \ldots + 6n = 3n(n+1)$ for all positive integers n.

22. Prove by induction that $1^4 + 2^4 + 3^4 + \ldots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$ for all positive integers n.

23. Prove by induction that $4 + 9 + 14 + \ldots + (5n-1) = \frac{n(5n+3)}{2}$ for all positive integers n.

24. Prove by induction that $7 + 14 + 21 + \ldots + 7n = \frac{7n(n+1)}{2}$ for all positive integers n.

25. Prove by induction that $1 + 6 + 11 + \ldots + (5n-4) = \frac{n(5n-3)}{2}$ for all positive integers n.

26. Let $s_1 = 5$ and $s_{n+1} = 2s_n + 3$ for $n \geq 1$. Prove by induction that $s_n = 2^{n+2} - 3$ for all positive integers n.

27. Prove by induction that $\sum_{r=1}^{n} r \cdot 5^r = \frac{(4n-1)5^{n+1}+5}{16}$ for all positive integers n.

28. Prove by induction that $\sum_{r=1}^{n} \frac{1}{r(r+3)} = \frac{11n+18}{18(n+1)(n+2)(n+3)} \cdot \frac{n(n+2)(n+3)}{2}$ for all positive integers n.

29. The Tribonacci sequence is defined by $T_1 = 1$, $T_2 = 1$, $T_3 = 2$, and $T_{n+1} = T_n + T_{n-1} + T_{n-2}$ for $n \geq 3$. Prove by induction that $T_1 + T_2 + \ldots + T_n = \frac{T_{n+3}-2}{2}$ for all $n \geq 1$.

30. Prove by induction that $\sum_{r=1}^{n} r(r+1)(r+2) = \frac{n(n+1)(n+2)(n+3)}{4}$ for all positive integers n.

Section D: Mathematical Induction - Inequalities

31. Prove by induction that $6^n \geq 5n + 1$ for all non-negative integers n.

32. Prove by induction that $4^n \geq 3n^2$ for all integers $n \geq 2$.

33. Prove by induction that $n! \geq 5^{n-4}$ for all integers $n \geq 6$.

34. Prove by induction that $(1-x)^n \leq \frac{1}{1+nx}$ for all real $x \geq 0$ and all positive integers n.

35. Prove by induction that $\frac{1}{4^2} + \frac{1}{5^2} + \ldots + \frac{1}{n^2} < \frac{1}{3} - \frac{1}{3n}$ for all integers $n \geq 4$.

36. Prove by induction that $\frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \ldots + \frac{1}{\sqrt{n}} \geq 2(\sqrt{n} - \sqrt{3})$ for all integers $n \geq 4$.

37. Prove by induction that $1 + \frac{1}{4^2} + \frac{1}{5^2} + \ldots + \frac{1}{n^2} < \frac{4}{3}$ for all integers $n \geq 4$.

38. Prove by induction that $5^n \geq 2n^3$ for all integers $n \geq 4$.

39. Prove by induction that $\left(1 + \frac{1}{4n}\right)^n < \frac{3}{2}$ for all positive integers n.

40. Prove by induction that for $n \geq 4$, $\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} \geq \frac{11}{20}$.

Section E: Mathematical Induction - Divisibility

41. Prove by induction that $n^3 + 17n$ is divisible by 6 for all positive integers n.

42. Prove by induction that $7^n - 1$ is divisible by 6 for all positive integers n.

43. Prove by induction that $9^n - 1$ is divisible by 8 for all positive integers n.

44. Prove by induction that $2n^3 + 3n^2 + n$ is divisible by 6 for all positive integers n.

45. Prove by induction that $11^n - 1$ is divisible by 10 for all positive integers n.

46. Prove by induction that $5^{2n+1} + 2^{n+2}$ is divisible by 7 for all non-negative integers n.

47. Prove by induction that $13^n - 6^n$ is divisible by 7 for all positive integers n.
48. Prove by induction that $7^{2n} - 1$ is divisible by 48 for all positive integers n.
49. Prove by induction that $n^{13} - n$ is divisible by 13 for all positive integers n.
50. Prove by induction that $15^n + 16^n$ is divisible by 31 for all odd positive integers n.

Section F: Deduction in Algebraic Manipulation

51. Given that $s + t = 10$ and $st = 21$, find the value of $s^3 + t^3$.
52. If $x + y + z = 4$ and $xy + yz + zx = 1$, find the value of $x^2 + y^2 + z^2$.
53. Given that α and β are roots of $x^2 - 4x + 2 = 0$, prove that:
    (a) $\alpha + \beta = 4$
    (b) $\alpha\beta = 2$
    (c) $\alpha^3 + \beta^3 = 52$
54. If $z + \frac{1}{z} = 5$, find expressions for:
    (a) $z^2 + \frac{1}{z^2}$
    (b) $z^3 + \frac{1}{z^3}$
    (c) $z^4 + \frac{1}{z^4}$
55. Prove that if $a + b + c = 0$, then $a^2(b + c) + b^2(c + a) + c^2(a + b) = -3abc$.
56. Given that m, n, p are in geometric progression, prove that $\log m + \log p = 2 \log n$.
57. If $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$, prove that $A + B = \frac{\pi}{4}$.
58. Prove that $(x + y - z)^2 + (y + z - x)^2 + (z + x - y)^2 = 2(x^2 + y^2 + z^2) - 2xy - 2yz - 2zx$.
59. Given that a, b, c are in arithmetic progression with common difference d, prove that $a^2 + c^2 = 2b^2 + 2d^2$.
60. If $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ and $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0$, prove that $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Section G: Deduction in Geometric Reasoning

61. In triangle XYZ, prove that the measure of an exterior angle equals 180 minus the measure of the adjacent interior angle.
62. Prove that if two circles touch internally, the line joining their centers passes through the point of contact.
63. Prove that the angle subtended by a chord at the center is twice the angle subtended by the same chord at any point on the major arc.
64. In triangle ABC, let O be the circumcenter. Prove that $\angle BOC = 2\angle A$ when A is acute.
65. Prove that if a parallelogram has one right angle, then it is a rectangle.
66. In a circle, prove that equal arcs subtend equal chords.
67. Prove that the tangent to a circle is perpendicular to the radius at the point of tangency.
68. In triangle PQR, prove that $p^2 = q^2 + r^2 - 2qr \cos P$ where p, q, r are the sides opposite to angles P, Q, R respectively.

69. Prove that the medians of a triangle intersect at a point that divides each median in the ratio $2 : 1$.

70. Prove that if the diagonals of a quadrilateral bisect each other at right angles, then it is a rhombus.

Section H: Advanced Proof Techniques

71. Prove that there exists a rational number between any two distinct irrational numbers.

72. Prove that if $h(x) = \frac{x-3}{2x+1}$ where $x \neq -\frac{1}{2}$, then h is bijective on its domain.

73. Prove that the set of negative integers has the same cardinality as the set of positive integers.

74. Use the pigeonhole principle to prove that among any 8 people, at least two were born on the same day of the week.

75. Prove that $3 + \sqrt{7}$ is irrational.

76. Prove that if $\gcd(a, b) = 1$ and c divides ab, then $\gcd(c, a)$ divides b and $\gcd(c, b)$ divides a.

77. Prove that if r is rational and s is irrational, then rs is irrational (provided $r \neq 0$).

78. Use strong induction to prove that every integer $n \geq 12$ can be expressed as $4a + 5b$ where a and b are non-negative integers.

79. Prove that if $a_1, a_2, \ldots, a_n$ are positive real numbers, then:

$$\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}} \leq \sqrt[n]{a_1 a_2 \cdots a_n}$$

(HM-GM inequality)

80. Prove or disprove: The expression $n^3 + n + 1$ is prime for all positive integers n.

Section I: Proof Writing and Communication

81. Write a complete proof that for any triangle with sides a, b, c and angles A, B, C opposite to these sides respectively, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

82. Prove that the equation $x^6 + y^6 = z^3$ has no positive integer solutions.

83. Let $G_n = 2^1 + 2^2 + 2^3 + \ldots + 2^n$. Prove that $G_n = 2^{n+1} - 2$ and use this to show that G_n is never divisible by 3 for $n \geq 1$.

84. Prove the Apollonius identity: For any quadrilateral with sides a, b, c, d and diagonals p, q:

$$p^2 + q^2 = a^2 + b^2 + c^2 + d^2 - 4m^2$$

where m is the distance between the midpoints of the diagonals.

85. Consider the sequence defined by $f_1 = 3$, $f_2 = 8$, and $f_{n+2} = f_{n+1} + f_n$ for $n \geq 1$. Prove that $\gcd(f_n, f_{n+1}) = 1$ for all $n \geq 1$.

86. Prove that for any positive integer n, the number $8^{2n} - 3^{n+1}$ is divisible by 13.

87. Let $\ell : \mathbb{R} \setminus \{3\} \to \mathbb{R} \setminus \{-2\}$ be defined by $\ell(x) = \frac{-2x+5}{x-3}$. Prove that ℓ is bijective and find its inverse function.

88. Prove the Chinese Remainder Theorem for two congruences: If $\gcd(m, n) = 1$, then the system $x \equiv a \pmod{m}$ and $x \equiv b \pmod{n}$ has a unique solution modulo mn.

89. Prove that $\log_a b$ is irrational when a and b are positive integers greater than 1 and b is not a perfect power of a.

90. Write a proof demonstrating that there are infinitely many primes of the form $4k + 3$ where k is a non-negative integer.

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

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