

A Level Pure Mathematics

Practice Test 4: Algebra and Functions

Instructions:

Answer all questions. Show your working clearly.

Calculators may be used unless stated otherwise.

Time allowed: 2 hours

Section A: Algebraic Manipulation

1. Simplify these expressions:

(a) $\frac{x^2-36}{x^2-12x+36}$

(b) $\frac{5x^2-20}{x^2+x-12}$

(c) $\frac{x^3+125}{x^2+10x+25}$

(d) $\frac{x^4-625}{x^2-5x-50}$

2. Factorize completely:

(a) $x^3 - 12x^2 + 48x - 64$

(b) $125x^3 - 8$

(c) $x^6 - 729$

(d) $x^{12} - 1$

(e) $x^4 + 12x^2 + 36$

(f) $x^3 + 4x^2 - 25x - 100$

3. Express as single fractions in simplest form:

(a) $\frac{5}{x+4} - \frac{3}{x-2}$

(b) $\frac{4x}{x^2-16} + \frac{3}{x-4}$

(c) $\frac{2x-3}{x^2+5x+6} - \frac{x+2}{x^2-9}$

(d) $\frac{4}{x-2} + \frac{3}{x+1} - \frac{5}{x^2-x-2}$

4. Use the binomial theorem to expand:

(a) $(5x + 2)^4$

(b) $(3x - \frac{1}{2x})^6$

(c) $(1 - 3x)^7$, and find the coefficient of x^5

(d) Find the middle term in the expansion of $(x^2 + \frac{3}{x})^{10}$

5. Simplify using laws of indices:

(a) $\frac{7^{3x-1} \cdot 49^{x+2}}{343^x}$

- (b) $\frac{64^x \cdot 32^{2x-1}}{128^{x+1}}$
- (c) $(x^{\frac{3}{7}})^{\frac{7}{9}} \cdot x^{-\frac{2}{3}}$
- (d) $\frac{(6x)^2 \cdot (2x^3)^3}{12x^8}$

Section B: Linear and Quadratic Equations

6. Solve these equations:

- (a) $\frac{5x-2}{6} + \frac{2x+3}{4} = \frac{3}{2}$
- (b) $\frac{4x}{x+3} = \frac{6}{x-1}$
- (c) $\sqrt{5x+1} = 3x-1$
- (d) $\frac{4}{x+3} + \frac{2}{x-2} = \frac{3}{5}$

7. Solve these quadratic equations, leaving answers in exact form where appropriate:

- (a) $6x^2 - 11x + 3 = 0$
- (b) $x^2 - 10x + 7 = 0$
- (c) $5x^2 = 8x + 3$
- (d) $(5x-2)^2 = 3(3x-1)$

8. For the quadratic equation $4x^2 - (3k-2)x + 2k = 0$:

- (a) Find the discriminant in terms of k
- (b) Find the values of k for which the equation has equal roots
- (c) Find the values of k for which the equation has no real roots
- (d) When $k = 4$, find the sum and product of the roots

9. The quadratic $rx^2 + sx + t = 0$ has roots α and β .

- (a) Express $\alpha + \beta$ and $\alpha\beta$ in terms of r , s , and t
- (b) Find a quadratic equation with roots $\alpha - 3$ and $\beta - 3$
- (c) Find a quadratic equation with roots $\frac{\alpha}{2}$ and $\frac{\beta}{2}$
- (d) If $\alpha^2 + \beta^2 = 22$ and $\alpha + \beta = 7$, find $\alpha\beta$

Section C: Cubic and Higher Order Equations

10. Solve these cubic equations:

- (a) $x^3 - 8x^2 + 19x - 12 = 0$
- (b) $x^3 - 4x^2 - 11x + 30 = 0$
- (c) $6x^3 + x^2 - 16x + 8 = 0$
- (d) $x^3 - 12x^2 + 47x - 60 = 0$

11. Given that $x = -2$ is a root of $x^3 + 3x^2 + ax + b = 0$:

- (a) Find a relationship between a and b
- (b) If the other two roots are equal, find a and b
- (c) Hence find all three roots
- (d) Express the cubic in factored form

12. Solve these quartic equations:

- (a) $x^4 - 20x^2 + 64 = 0$
 - (b) $x^4 - 7x^2 + 12 = 0$
 - (c) $(x^2 + 2x)^2 - 11(x^2 + 2x) + 24 = 0$
 - (d) $x^4 + x^3 - 9x^2 - 3x + 18 = 0$ (given that $x = -3$ is a root)
13. Use the substitution $v = x - \frac{3}{x}$ to solve:
- (a) $x^2 + \frac{9}{x^2} = 11$
 - (b) $5x^2 - 2x + \frac{6}{x} - \frac{15}{x^2} = 0$

Section D: Functions - Definition and Notation

14. For the function $f(x) = \frac{5x+1}{3x-2}$ where $x \neq \frac{2}{3}$:
- (a) Find $f(0)$, $f(1)$, and $f(-1)$
 - (b) Solve $f(x) = 4$
 - (c) Find the value of x for which $f(x)$ is undefined
 - (d) Find the range of $f(x)$
15. Given $g(x) = x^2 + 10x + 21$:
- (a) Express $g(x)$ in the form $(x + p)^2 + q$
 - (b) State the minimum value of $g(x)$ and the value of x at which it occurs
 - (c) Solve $g(x) = 0$
 - (d) Find the range of $g(x)$
16. For $h(x) = \sqrt{36 - x^2}$:
- (a) Find the domain of $h(x)$
 - (b) Find the range of $h(x)$
 - (c) Sketch the graph of $y = h(x)$
 - (d) Solve $h(x) = 5$
17. Define $k(x) = \begin{cases} 3x^2 - 1 & \text{if } x \leq 1 \\ 2x + 1 & \text{if } 1 < x < 4 \\ 9 & \text{if } x \geq 4 \end{cases}$
- (a) Find $k(-1)$, $k(1)$, $k(2.5)$, and $k(5)$
 - (b) Is $k(x)$ continuous at $x = 1$? Justify your answer
 - (c) Is $k(x)$ continuous at $x = 4$? Justify your answer
 - (d) Sketch the graph of $y = k(x)$

Section E: Composite and Inverse Functions

18. Given $f(x) = 6x - 2$ and $g(x) = x^2 + 4$:
- (a) Find $f(g(x))$ and $g(f(x))$
 - (b) Calculate $f(g(0))$ and $g(f(0))$
 - (c) Solve $f(g(x)) = 28$
 - (d) Find $(f \circ g)^{-1}(x)$

19. For $p(x) = \frac{4x-3}{2x+1}$ where $x \neq -\frac{1}{2}$:
- (a) Find $p^{-1}(x)$
 - (b) Verify that $p(p^{-1}(x)) = x$
 - (c) State the domain and range of $p^{-1}(x)$
 - (d) Solve $p(x) = p^{-1}(x)$
20. Given $f(x) = 7x + 1$ and $g(x) = \frac{3}{x-1}$ where $x \neq 1$:
- (a) Find $(f \circ g)(x)$ and state its domain
 - (b) Find $(g \circ f)(x)$ and state its domain
 - (c) Find $(f \circ g)^{-1}(x)$
 - (d) Verify your answer by showing $(f \circ g)((f \circ g)^{-1}(x)) = x$
21. The function $h(x) = x^2 - 12x + 5$ is defined for $x \geq 6$.
- (a) Explain why the domain restriction is necessary for h^{-1} to exist
 - (b) Find $h^{-1}(x)$
 - (c) State the domain and range of $h^{-1}(x)$
 - (d) Sketch $h(x)$ and $h^{-1}(x)$ on the same axes

Section F: Graphing Functions

22. Sketch the graphs of these functions, clearly showing key features:

- (a) $y = x^3 + 6x^2 + 12x + 8$
- (b) $y = \frac{5x+1}{3x-2}$
- (c) $y = |x^2 - 10x + 21|$
- (d) $y = \frac{x^2+16}{x^2-16}$

23. For the rational function $f(x) = \frac{x^2+4x+3}{x^2-16}$:

- (a) Find the domain of $f(x)$
- (b) Find the x and y intercepts
- (c) Identify any vertical asymptotes
- (d) Find the horizontal asymptote
- (e) Sketch the graph of $y = f(x)$

24. Analyze the function $g(x) = \frac{6x^2-24}{x^2+2x-15}$:

- (a) Factorize the numerator and denominator
- (b) Simplify $g(x)$ and state its domain
- (c) Find any asymptotes
- (d) Find the coordinates of any stationary points
- (e) Sketch the graph of $y = g(x)$

25. For the polynomial $p(x) = x^4 - 8x^3 + 16x^2$:

- (a) Factorize $p(x)$ completely
- (b) Find the roots and their multiplicities
- (c) Determine the behavior at each root
- (d) Find $p'(x)$ and locate stationary points
- (e) Sketch the graph of $y = p(x)$

Section G: Function Transformations

26. Given the function $f(x) = x^2$, describe the transformations and sketch:
- (a) $y = f(x + 1) - 3$
 - (b) $y = -4f(x - 2)$
 - (c) $y = f(5x) + 2$
 - (d) $y = \frac{3}{2}f(-x) - 1$
27. The graph of $y = f(x)$ has vertex at $(4, -1)$ and passes through $(2, 3)$ and $(6, 3)$. Find the vertex and two other points for:
- (a) $y = f(x) - 5$
 - (b) $y = f(x + 2)$
 - (c) $y = 4f(x)$
 - (d) $y = f(\frac{x}{2})$
 - (e) $y = -f(x)$
 - (f) $y = f(-x)$
28. Given that $g(x) = |x + 4| - 2$:
- (a) Describe the transformations applied to $y = |x|$
 - (b) State the vertex of the graph
 - (c) Find the range of $g(x)$
 - (d) Solve $g(x) = 4$
 - (e) Sketch the graph of $y = g(x)$
29. The function $h(x) = \cot x$ is transformed to $k(x) = 3 \cot(4x + \pi) + 2$.
- (a) Identify each transformation in the correct order
 - (b) State the period of $k(x)$
 - (c) Find the phase shift
 - (d) Find the vertical shift
 - (e) Find the vertical asymptotes in the interval $[0, \pi]$
 - (f) Sketch one complete cycle of $y = k(x)$

Section H: Special Functions and Applications

30. For the exponential function $f(x) = 4^{x+1} - 5$:
- (a) State the domain and range
 - (b) Find the y-intercept
 - (c) Find the horizontal asymptote
 - (d) Solve $f(x) = 59$
 - (e) Find $f^{-1}(x)$ and state its domain and range
31. For the logarithmic function $g(x) = \log_5(3x + 2) - 2$:
- (a) State the domain and range
 - (b) Find the x-intercept
 - (c) Find the vertical asymptote

- (d) Solve $g(x) = 3$
- (e) Express $g(x)$ in terms of natural logarithms
32. A function is defined as $f(x) = \frac{ux+v}{wx+y}$ where $uy - vw \neq 0$.
- (a) Find the domain of $f(x)$
- (b) Find $f^{-1}(x)$
- (c) Show that $(f^{-1} \circ f)(x) = x$
- (d) Find the condition under which f is an involution (i.e., $f(f(x)) = x$)
33. The modulus function $|x|$ can be written as: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$
- (a) Sketch $y = |5x - 2|$
- (b) Solve $|5x - 2| = 8$
- (c) Solve $|5x - 2| > 7$
- (d) Find the range of values for which $|5x - 2| \leq 3$

Section I: Problem Solving and Applications

34. A semicircular arch is to be constructed with a perimeter of 100 meters. Let r be the radius.
- (a) Express the perimeter in terms of r
- (b) Show that the area $A = r(50 - \frac{\pi r}{2})$
- (c) Find the value of r that maximizes the area
- (d) Calculate the maximum area
- (e) State the domain of the function in this context
35. The height h (in meters) of a rocket after t seconds is given by: $h(t) = -4t^2 + 32t + 12$
- (a) Express $h(t)$ in completed square form
- (b) Find when the rocket reaches maximum height
- (c) Calculate the maximum height
- (d) Determine when the rocket returns to ground level
- (e) Find the rocket's height after 5 seconds
36. A company's daily production output O (in units) depends on the number of workers w according to: $O(w) = -2w^2 + 24w + 50$ for $1 \leq w \leq 15$
- (a) Find the number of workers that maximizes output
- (b) Calculate the maximum daily output
- (c) Determine how many workers are needed for an output of 122 units
- (d) Find the output when there are 8 workers
37. A function $f(x) = \frac{x^2+9}{x^2-25}$ models a response ratio.
- (a) Find the domain and range of $f(x)$
- (b) Determine any asymptotes and explain their significance
- (c) Find when $f(x) = 1$
- (d) Analyze the behavior as $x \rightarrow \pm\infty$
- (e) Sketch the graph and identify any symmetry

38. Two functions are related by $g(x) = f(5x + 1) - 4$ where $f(x) = x^2$.

- (a) Find an explicit expression for $g(x)$
- (b) Describe the transformations that map f to g
- (c) Find the vertex of the parabola $y = g(x)$
- (d) If f has domain $[-4, 1]$, find the domain of g
- (e) Solve $g(x) = f(x)$ and describe the intersection points

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

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