A Level Pure Mathematics
Practice Test 3: Vectors

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Vector Basics and Notation

1. Given vectors $\mathbf{p} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$, calculate:

   (a) $\mathbf{p} + \mathbf{q}$

   (b) $\mathbf{p} - \mathbf{q}$

   (c) $2\mathbf{p} + 3\mathbf{q}$

   (d) $5\mathbf{p} - 2\mathbf{q}$

   (e) $|\mathbf{p}|$ and $|\mathbf{q}|$

   (f) A unit vector in the direction of $\mathbf{q}$

2. Express these vectors in component form:

   (a) $\overrightarrow{EF}$ where $E(1, 4, -1)$ and $F(3, 2, 5)$

   (b) $\overrightarrow{GH}$ where $G(-3, 2, 4)$ and $H(2, -1, 3)$

   (c) The position vector of point K if $\overrightarrow{OK} = 3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$

   (d) $\overrightarrow{FE}$ where $E(4, -2, 3)$ and $F(2, 1, -5)$

3. Given $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$:

   (a) Find $|\mathbf{a}|$ and $|\mathbf{b}|$

   (b) Calculate $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$

   (c) Find scalars r and s such that $r\mathbf{a} + s\mathbf{b} = \begin{pmatrix} 2 \\ -5 \\ 8 \end{pmatrix}$

   (d) Determine if $\mathbf{a}$ and $\mathbf{b}$ are parallel

4. Points X, Y, and Z have position vectors $\mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$, and $\mathbf{z} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$.

   (a) Find vectors $\overrightarrow{XY}$ and $\overrightarrow{XZ}$

   (b) Calculate the lengths $|XY|$ and $|XZ|$

(c) Find the position vector of the midpoint of YZ

(d) Determine if triangle XYZ is isosceles

5. Find the values of m for which these vectors are perpendicular:

(a) $\mathbf{u} = \begin{pmatrix} 2 \\ m \\ 4 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} m \\ 3 \\ -2 \end{pmatrix}$

(b) $\mathbf{r} = \begin{pmatrix} 3 \\ 2m \\ 1 \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} 2 \\ -1 \\ m \end{pmatrix}$

(c) $\mathbf{t} = m\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ and $\mathbf{w} = 3\mathbf{i} + m\mathbf{j} + 2\mathbf{k}$

Section B: Dot Product (Scalar Product)

6. Calculate the dot product of these vectors:

(a) $\mathbf{c} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$

(b) $\mathbf{e} = 3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and $\mathbf{f} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

(c) $\mathbf{g} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{h} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$

(d) $\mathbf{k} = 4\mathbf{i} + 2\mathbf{j}$ and $\mathbf{l} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

7. Find the angle between these pairs of vectors:

(a) $\mathbf{m} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

(b) $\mathbf{p} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$

(c) $\mathbf{r} = 4\mathbf{i} + 3\mathbf{j}$ and $\mathbf{s} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

(d) $\mathbf{t} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

8. Use the dot product to verify these properties:

(a) $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$ (commutative)

(b) $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$ (distributive)

(c) $(k\mathbf{x}) \cdot \mathbf{y} = k(\mathbf{x} \cdot \mathbf{y})$ for scalar k

(d) $\mathbf{x} \cdot \mathbf{x} = |\mathbf{x}|^2$

9. Given vectors $\mathbf{p} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$, and $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$:

(a) Show that $\mathbf{p}$ and $\mathbf{q}$ are perpendicular

(b) Find the component of $\mathbf{r}$ in the direction of $\mathbf{p}$

(c) Calculate $|\mathbf{p} + \mathbf{q} + \mathbf{r}|$

(d) Find the angle between $\mathbf{p}+\mathbf{q}$ and $\mathbf{r}$

10. A triangle has vertices at $L(3,2,1)$, $M(1,4,3)$, and $N(2,1,4)$.
    (a) Find the vectors $\overrightarrow{LM}$ and $\overrightarrow{LN}$
    (b) Calculate the angle $\angle MLN$
    (c) Find the area of triangle LMN
    (d) Determine if the triangle is right-angled

Section C: Cross Product (Vector Product)

11. Calculate the cross product of these vectors:
    (a) $\mathbf{p} = \begin{pmatrix}3\\2\\1\end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix}1\\3\\4\end{pmatrix}$
    (b) $\mathbf{r} = 4\mathbf{i}+\mathbf{j}-2\mathbf{k}$ and $\mathbf{s} = 2\mathbf{i}+3\mathbf{j}+\mathbf{k}$
    (c) $\mathbf{t} = \begin{pmatrix}2\\-3\\1\end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix}1\\2\\-4\end{pmatrix}$
    (d) $\mathbf{v} = 3\mathbf{i}+4\mathbf{j}$ and $\mathbf{w} = 2\mathbf{i}+5\mathbf{k}$

12. Verify these properties of the cross product:
    (a) $\mathbf{x}\times\mathbf{y} = -(\mathbf{y}\times\mathbf{x})$ (anti-commutative)
    (b) $\mathbf{x}\times(\mathbf{y}+\mathbf{z}) = \mathbf{x}\times\mathbf{y}+\mathbf{x}\times\mathbf{z}$ (distributive)
    (c) $\mathbf{x}\times\mathbf{x} = \mathbf{0}$
    (d) $|\mathbf{x}\times\mathbf{y}|^2 = |\mathbf{x}|^2|\mathbf{y}|^2-(\mathbf{x}\cdot\mathbf{y})^2$

13. Find the area of the parallelogram spanned by:
    (a) $\mathbf{a} = \begin{pmatrix}2\\3\\0\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}1\\2\\4\end{pmatrix}$
    (b) $\mathbf{c} = 4\mathbf{i}+2\mathbf{j}-3\mathbf{k}$ and $\mathbf{d} = 2\mathbf{i}-\mathbf{j}+4\mathbf{k}$
    (c) Vectors from origin to points $(4,2,1)$ and $(1,3,2)$
    (d) $\overrightarrow{ST}$ and $\overrightarrow{SU}$ where $S(3,1,2)$, $T(2,4,1)$, $U(1,2,4)$

14. Given $\mathbf{a} = \begin{pmatrix}4\\-1\\2\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}1\\3\\-4\end{pmatrix}$:
    (a) Calculate $\mathbf{a}\times\mathbf{b}$
    (b) Verify that $\mathbf{a}\times\mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$
    (c) Find a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$
    (d) Calculate the area of triangle with sides $\mathbf{a}$ and $\mathbf{b}$

15. Use the scalar triple product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ to find:
    (a) The volume of parallelepiped with edges $\mathbf{a} = \begin{pmatrix}3\\1\\2\end{pmatrix}$, $\mathbf{b} = \begin{pmatrix}1\\3\\1\end{pmatrix}$, $\mathbf{c} = \begin{pmatrix}2\\1\\3\end{pmatrix}$
    (b) Whether points $A(3,1,2)$, $B(2,4,1)$, $C(1,2,4)$, $D(3,3,3)$ are coplanar
    (c) The volume of tetrahedron with vertices at $(0,0,0)$, $(3,1,2)$, $(1,3,1)$, $(2,1,3)$

Section D: Equations of Lines

16. Find the vector equation of the line:

    (a) Passing through $Q(3,1,4)$ in direction $\begin{pmatrix}2\\-3\\1\end{pmatrix}$

    (b) Passing through points $R(2,4,1)$ and $S(1,2,5)$

    (c) Through origin parallel to vector $4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

    (d) Through $(2,3,4)$ parallel to the line $\mathbf{r} = \begin{pmatrix}1\\4\\2\end{pmatrix} + t\begin{pmatrix}3\\1\\-4\end{pmatrix}$

17. Convert these to parametric form:

    (a) $\mathbf{r} = \begin{pmatrix}3\\2\\1\end{pmatrix} + t\begin{pmatrix}1\\-3\\2\end{pmatrix}$

    (b) $\mathbf{r} = \begin{pmatrix}2\\1\\4\end{pmatrix} + s\begin{pmatrix}3\\2\\1\end{pmatrix}$

    (c) Line through $(1,4,2)$ and $(3,2,1)$

    (d) $\mathbf{r} = (1+4t)\mathbf{i} + (3-2t)\mathbf{j} + (2+3t)\mathbf{k}$

18. Find where these lines intersect the coordinate planes:

    (a) $\mathbf{r} = \begin{pmatrix}3\\2\\6\end{pmatrix} + t\begin{pmatrix}1\\2\\-3\end{pmatrix}$ and the xy-plane

    (b) $\mathbf{r} = \begin{pmatrix}2\\4\\1\end{pmatrix} + s\begin{pmatrix}3\\-2\\2\end{pmatrix}$ and the xz-plane

    (c) Line through $(4,3,2)$ and $(2,1,0)$ with the yz-plane

19. Determine if these pairs of lines intersect, are parallel, or are skew:

    (a) $L_1 : \mathbf{r} = \begin{pmatrix}3\\1\\4\end{pmatrix} + t\begin{pmatrix}1\\3\\2\end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix}2\\4\\1\end{pmatrix} + s\begin{pmatrix}2\\1\\4\end{pmatrix}$

    (b) $L_1 : \mathbf{r} = \begin{pmatrix}1\\3\\0\end{pmatrix} + t\begin{pmatrix}2\\1\\4\end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix}2\\1\\3\end{pmatrix} + s\begin{pmatrix}4\\2\\8\end{pmatrix}$

    (c) Lines through $(3,2,1)$ to $(1,4,2)$ and $(2,1,4)$ to $(4,3,1)$

20. Find the shortest distance between:

    (a) Point $(2,4,1)$ and line $\mathbf{r} = \begin{pmatrix}3\\1\\2\end{pmatrix} + t\begin{pmatrix}1\\3\\2\end{pmatrix}$

    (b) Parallel lines $L_1 : \mathbf{r} = \begin{pmatrix}3\\2\\1\end{pmatrix} + t\begin{pmatrix}2\\1\\3\end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix}1\\4\\2\end{pmatrix} + s\begin{pmatrix}2\\1\\3\end{pmatrix}$

    (c) Skew lines $L_1 : \mathbf{r} = \begin{pmatrix}2\\1\\0\end{pmatrix} + t\begin{pmatrix}3\\1\\2\end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix}1\\0\\3\end{pmatrix} + s\begin{pmatrix}1\\2\\0\end{pmatrix}$

Section E: Equations of Planes

21. Find the equation of the plane:

    (a) With normal vector $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ passing through $(1, 4, 2)$

    (b) Passing through points $(3, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 4)$

    (c) Containing the lines $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} + s \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

    (d) Parallel to vectors $\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ through $(1, 3, 3)$

22. Convert between vector and Cartesian forms:

    (a) $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ to Cartesian form

    (b) $2x - 4y + 3z = 12$ to vector form

    (c) $3x + 2y - z = 9$ to parametric form

    (d) $\mathbf{r} \cdot \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = 6$ to Cartesian form

23. Find where these planes intersect coordinate axes:

    (a) $4x + 3y - 2z = 12$

    (b) $2x - 4y + z = 8$

    (c) $3x + 2y + 4z = 24$

    (d) $2x + 3y + z = 6$

24. Determine the relationship between these planes:

    (a) $\Pi_1 : 3x + 2y - z = 6$ and $\Pi_2 : 6x + 4y - 2z = 12$

    (b) $\Pi_1 : 2x - 3y + z = 4$ and $\Pi_2 : x + 2y - 3z = 6$

    (c) $\Pi_1 : 4x + 2y + z = 8$ and $\Pi_2 : 8x + 4y + 2z = 12$

    (d) $\Pi_1 : 3x - y + 2z = 9$ and $\Pi_2 : 2x + 3y - z = 6$

25. Find the line of intersection of these planes:

    (a) $3x + y + 2z = 7$ and $x - 2y + 3z = 4$

    (b) $2x + 3y - z = 8$ and $x - 2y + 4z = 1$

    (c) $4x - y + 2z = 6$ and $2x + 3y - z = 3$

    (d) $x + 4y + 2z = 10$ and $3x - 2y + z = 8$

Section F: Angles and Distances

26. Find the angle between these planes:

    (a) $4x + 2y - 3z = 6$ and $2x - 4y + z = 8$

    (b) $3x + 4y - z = 9$ and $2x - 3y + 4z = 7$

(c) $\mathbf{r} \cdot \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = 5$ and $\mathbf{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 6$

(d) $4x + y + 3z = 12$ and $2x - 4y + z = 8$

27. Calculate the distance from point to plane:

    (a) Point $(3, 2, 4)$ to plane $4x + 3y - 2z = 6$
    (b) Point $(2, -3, 1)$ to plane $3x - 2y + 4z = 9$
    (c) Point $(0, 0, 0)$ to plane $4x + 3y - 2z = 20$
    (d) Point $(4, 1, -2)$ to plane $2x - 4y + 3z = 10$

28. Find the angle between line and plane:

    (a) Line $\mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ and plane $4x + 2y + z = 9$
    (b) Line through $(3, 1, 4)$ and $(2, 5, 1)$ with plane $3x - 2y + 4z = 8$
    (c) Line $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$ and plane $2x + 4y - z = 6$

29. Determine where these lines intersect planes:

    (a) $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ and $x + 3y + 2z = 15$
    (b) Line through $(2, 4, 1)$ and $(1, 2, 4)$ with plane $3x - 2y + z = 6$
    (c) $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + s \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$ and $2x - 3y + 4z = 18$

30. Find the reflection of point in plane:

    (a) Point $(4, 2, 1)$ in plane $3x + 2y - z = 6$
    (b) Point $(1, -3, 4)$ in plane $2x - y + 4z = 10$
    (c) Point $(2, 4, 0)$ in plane $3x + y + 2z = 8$

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

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