A Level Pure Mathematics
Practice Test 3: Proof

Instructions:
Answer all questions. Show your working clearly.
Calculators may NOT be used in this test.
Time allowed: 2 hours

Section A: Direct Proof

1. Prove that the difference of two even integers is always even.
2. Prove that if n is an odd integer, then n^3 is odd.
3. Prove that the sum of three consecutive integers is always divisible by 3.
4. Prove that for any integer n, the expression $2n(n-1)$ is always even.
5. Given that r and s are rational numbers, prove that rs is rational.
6. Prove that if $a \geq 0$ and $b \geq 0$, then $\frac{2ab}{a+b} \leq \sqrt{ab}$ (HM-GM inequality).
7. Prove that for any real numbers p and q, $p^2 + q^2 \geq 2|pq|$.
8. Prove that if x, y, and z are the sides of a triangle, then $x + y \geq z$, $y + z \geq x$, and $z + x \geq y$.
9. Let $m(x) = x^7 - 5x^5 + 3x^3 - x$. Prove that m is an odd function.
10. Prove that the function $n(x) = -2x + 5$ is strictly decreasing on $\mathbb{R}$.

Section B: Proof by Contradiction

11. Prove that $\sqrt{11}$ is irrational.
12. Prove that there are infinitely many even numbers.
13. Prove that $\sqrt{8}$ is irrational.
14. Prove that if n^2 is divisible by 3, then n is divisible by 3.
15. Prove that there is no smallest positive real number.
16. Prove that if a and b are integers with $a^2 + b^2 = 6$, then at least one of a or b is even.
17. Prove that $\log_7 5$ is irrational.
18. Prove that the equation $x^2 + x + 1 = 0$ has no real solutions.
19. Prove that the equation $x^2 - 7y^2 = 4$ has no integer solutions.
20. Prove that if n is an integer and n^2 is odd, then n is odd.

Section C: Mathematical Induction - Sequences and Series

21. Prove by induction that $4 + 8 + 12 + \ldots + 4n = 2n(n + 1)$ for all positive integers n.
22. Prove by induction that $2^2 + 4^2 + 6^2 + \ldots + (2n)^2 = \frac{2n(n+1)(2n+1)}{3}$ for all positive integers n.
23. Prove by induction that $3 + 7 + 11 + \ldots + (4n - 1) = n(2n + 1)$ for all positive integers n.
24. Prove by induction that $1 + 8 + 15 + \ldots + (7n - 6) = \frac{n(7n-5)}{2}$ for all positive integers n.
25. Prove by induction that $5 + 9 + 13 + \ldots + (4n + 1) = n(2n + 3)$ for all positive integers n.
26. Let $z_1 = 1$ and $z_{n+1} = 3z_n + 2$ for $n \geq 1$. Prove by induction that $z_n = 3^n - 2$ for all positive integers n.
27. Prove by induction that $\sum_{r=1}^{n} r \cdot 4^r = \frac{(3n-1)4^{n+1}+4}{9}$ for all positive integers n.
28. Prove by induction that $\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$ for all positive integers n.
29. The Pell sequence is defined by $P_1 = 1$, $P_2 = 2$, and $P_{n+1} = 2P_n + P_{n-1}$ for $n \geq 2$. Prove by induction that $P_1 + P_2 + \ldots + P_n = \frac{P_{n+2}-2}{3}$ for all $n \geq 1$.
30. Prove by induction that $\sum_{r=1}^{n} r^2(r + 2) = \frac{n(n+1)(n+2)(3n+7)}{12}$ for all positive integers n.

Section D: Mathematical Induction - Inequalities

31. Prove by induction that $5^n \geq 4n + 1$ for all non-negative integers n.
32. Prove by induction that $3^n \geq 2n^2$ for all integers $n \geq 3$.
33. Prove by induction that $n! \geq 4^{n-3}$ for all integers $n \geq 5$.
34. Prove by induction that $(1 + x)^n \geq 1 + nx + \frac{n(n-1)}{2}x^2$ for all real $x \geq 0$ and all integers $n \geq 2$.
35. Prove by induction that $\frac{1}{3^2} + \frac{1}{4^2} + \ldots + \frac{1}{n^2} < \frac{1}{2} - \frac{1}{2n}$ for all integers $n \geq 3$.
36. Prove by induction that $\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots + \frac{1}{\sqrt{n}} \geq 2(\sqrt{n} - \sqrt{2})$ for all integers $n \geq 3$.
37. Prove by induction that $1 + \frac{1}{2^3} + \frac{1}{3^3} + \ldots + \frac{1}{n^3} < \frac{5}{4}$ for all positive integers n.
38. Prove by induction that $4^n \geq n^3$ for all integers $n \geq 6$.
39. Prove by induction that $\left(1 + \frac{1}{3n}\right)^n < 2$ for all positive integers n.
40. Prove by induction that for $n \geq 3$, $\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} \geq \frac{7}{12}$.

Section E: Mathematical Induction - Divisibility

41. Prove by induction that $n^3 + 11n$ is divisible by 6 for all positive integers n.
42. Prove by induction that $6^n - 1$ is divisible by 5 for all positive integers n.
43. Prove by induction that $8^n - 1$ is divisible by 7 for all positive integers n.
44. Prove by induction that $n^3 - n$ is divisible by 6 for all positive integers n.
45. Prove by induction that $10^n - 1$ is divisible by 9 for all positive integers n.
46. Prove by induction that $4^{2n} + 15n - 1$ is divisible by 9 for all positive integers n.
47. Prove by induction that $12^n - 5^n$ is divisible by 7 for all positive integers n.

48. Prove by induction that $5^{2n} - 1$ is divisible by 24 for all positive integers n.

49. Prove by induction that $n^{11} - n$ is divisible by 11 for all positive integers n.

50. Prove by induction that $14^n - 1$ is divisible by 13 for all positive integers n.

Section F: Deduction in Algebraic Manipulation

51. Given that $u + v = 8$ and $uv = 15$, find the value of $u^2 + v^2$.

52. If $p + q + r = 2$ and $pq + qr + rp = -3$, find the value of $p^2 + q^2 + r^2$.

53. Given that α and β are roots of $x^2 + 2x - 5 = 0$, prove that:
    (a) $\alpha + \beta = -2$
    (b) $\alpha\beta = -5$
    (c) $\alpha^2 + \beta^2 = 14$

54. If $y - \frac{1}{y} = 2$, find expressions for:
    (a) $y^2 + \frac{1}{y^2}$
    (b) $y^3 - \frac{1}{y^3}$
    (c) $y^4 + \frac{1}{y^4}$

55. Prove that if $x + y + z = 0$, then $x^3 + y^3 + z^3 = 3xyz$.

56. Given that a, b, c are in arithmetic progression, prove that $3b = a + c + b$.

57. If $\sin\theta + \sin\phi + \sin\psi = 0$ and $\cos\theta + \cos\phi + \cos\psi = 0$, prove that $\sin 3\theta + \sin 3\phi + \sin 3\psi = 3\sin(\theta + \phi + \psi)$.

58. Prove that $(p - q)^3 + (q - r)^3 + (r - p)^3 = 3(p - q)(q - r)(r - p)$.

59. Given that $\log x$, $\log y$, $\log z$ are in arithmetic progression, prove that $y^2 = xz$.

60. If p, q, r are in harmonic progression, prove that $\frac{2}{q} = \frac{1}{p} + \frac{1}{r}$.

Section G: Deduction in Geometric Reasoning

61. In triangle PQR, prove that each exterior angle equals the sum of the two non-adjacent interior angles.

62. Prove that the line joining the centers of two intersecting circles is perpendicular to their common chord.

63. Prove that opposite angles of a cyclic quadrilateral sum to 180.

64. In triangle ABC, let I be the incenter. Prove that $\angle BIC = 90 + \frac{\angle A}{2}$.

65. Prove that if two triangles have two sides and the included angle equal, then the triangles are congruent (SAS).

66. In a circle, prove that the angle between two chords equals half the sum of the intercepted arcs.

67. Prove that if two tangents are drawn to a circle from an external point, they make equal angles with the line joining that point to the center.

68. In triangle DEF, prove that $\frac{d}{\sin D} = \frac{e}{\sin E}$ where d and e are the sides opposite to angles D and E respectively.

69. Prove that the three angle bisectors of a triangle are concurrent at the incenter.
70. Prove that if a quadrilateral has one pair of opposite sides equal and parallel, then it is a parallelogram.

Section H: Advanced Proof Techniques

71. Prove that between any two distinct real numbers, there exists an irrational number.
72. Prove that if $g(x) = \frac{2x+3}{x-1}$ where $x \neq 1$, then g has an inverse function on its domain.
73. Prove that the set of odd positive integers has the same cardinality as the set of positive integers.
74. Use the pigeonhole principle to prove that in any set of 7 integers, at least two have the same remainder when divided by 6.
75. Prove that $2 + \sqrt{3}$ is irrational.
76. Prove that if p is prime and $p \geq 3$, then p is of the form $6k \pm 1$ for some integer k.
77. Prove that if a is rational and b is irrational, then $a + b$ is irrational (assuming $a \neq 0$).
78. Use strong induction to prove that every positive integer greater than 1 is either prime or can be written as a product of primes.
79. Prove that if $x_1, x_2, \ldots, x_n$ are positive real numbers, then:
$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \ldots + x_n}{n}$$
(GM-AM inequality)
80. Prove or disprove: For all positive integers n, $2^n - 1$ is prime.

Section I: Proof Writing and Communication

81. Write a complete proof that in any triangle with sides a, b, c and area S, the radius of the circumcircle is $R = \frac{abc}{4S}$.
82. Prove that the Diophantine equation $x^4 - y^4 = z^2$ has no positive integer solutions.
83. Let $B_n = \sum_{k=1}^{n} \frac{1}{k}$ be the n-th harmonic number. Prove that B_n is never an integer for $n \geq 2$.
84. Prove the parallelogram law: For vectors $\mathbf{u}$ and $\mathbf{v}$:
$$|\mathbf{u} + \mathbf{v}|^2 + |\mathbf{u} - \mathbf{v}|^2 = 2(|\mathbf{u}|^2 + |\mathbf{v}|^2)$$
85. Consider the sequence defined by $e_1 = 2$, $e_2 = 7$, and $e_{n+2} = e_{n+1} + e_n$ for $n \geq 1$. Prove that $\gcd(e_n, e_{n+1}) = 1$ for all $n \geq 1$.
86. Prove that for any positive integer n, the number $7^{2n} - 2^{3n}$ is divisible by 5.
87. Let $k : \mathbb{R} \setminus \{-2\} \to \mathbb{R} \setminus \{3\}$ be defined by $k(x) = \frac{3x+1}{x+2}$. Prove that k is bijective and find k^{-1}.
88. Prove Bézout's identity: For integers a and b with $\gcd(a, b) = d$, there exist integers x and y such that $ax + by = d$.
89. Prove that $\sqrt[3]{2}$ is irrational using the fundamental theorem of arithmetic.
90. Write a constructive proof showing that for any two distinct rational numbers r and s, there exists a rational number t such that $r < t < s$.

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

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