

A Level Pure Mathematics

Practice Test 4: Numerical Methods

Instructions:

Answer all questions. Show your working clearly.

Calculators may be used unless stated otherwise.

Time allowed: 2 hours

Section A: Introduction to Numerical Methods

1. Explain why numerical methods are needed for the following equations:

(a) $x^3 + 6x - 9 = 0$

(b) $x = \sec x$

(c) $e^x = 6x$

(d) $x^5 - 5x^3 + 3x - 5 = 0$

2. For each function, determine the approximate location of roots by examining sign changes:

(a) $f(x) = x^3 - 6x - 5$ for $x \in [-3, 4]$

(b) $f(x) = x^2 - 7x - 3$ for $x \in [-2, 8]$

(c) $f(x) = e^x - 5x - 1$ for $x \in [-1, 3]$

(d) $f(x) = \ln x - x + 5$ for $x \in [4, 7]$

3. State the conditions required for the Intermediate Value Theorem and explain how it guarantees the existence of roots.

4. For the function $f(x) = x^3 - 4x - 4$:

(a) Show that there is a root between $x = 2$ and $x = 3$

(b) Determine a more precise interval containing the root

(c) Sketch the graph of $y = f(x)$ showing the root location

(d) Explain why this equation cannot be solved algebraically

5. Define the following terms in the context of numerical methods:

(a) Absolute error

(b) Relative error

(c) Tolerance

(d) Convergence

(e) Iteration

(f) Fixed point

Section B: Bisection Method

6. Use the bisection method to find the root of $f(x) = x^3 - 6x - 9$ in the interval $[2, 3]$.
 - (a) Complete 4 iterations
 - (b) Give your answer correct to 2 decimal places
 - (c) Estimate the error in your final approximation
 - (d) How many iterations would be needed for an accuracy of 10^{-8} ?
7. Apply the bisection method to solve $x = \sec x$:
 - (a) Show that a root lies between 1 and 1.5
 - (b) Perform 5 iterations starting with $[1, 1.5]$
 - (c) Give your answer to 4 decimal places
 - (d) Verify your answer by substitution
8. Use the bisection method to find the positive root of $x^2 - 8x - 4 = 0$:
 - (a) Determine a suitable starting interval
 - (b) Perform iterations until the root is accurate to 3 decimal places
 - (c) Compare with the exact solution using the quadratic formula
 - (d) Calculate the absolute error
9. For the equation $e^x = 6x$:
 - (a) Show graphically that there are two roots
 - (b) Use bisection to find the smaller positive root to 3 decimal places
 - (c) Find the larger root to 3 decimal places
 - (d) Discuss the convergence rate of the bisection method
10. The equation $\ln x = 6 - x$ has a root near $x = 5$.
 - (a) Use bisection method with initial interval $[4, 6]$
 - (b) Continue until consecutive approximations differ by less than 0.01
 - (c) How many iterations were required?
 - (d) What is the theoretical minimum number of iterations needed?

Section C: Newton-Raphson Method

11. Use the Newton-Raphson method to solve $x^3 - 6x - 9 = 0$ starting with $x_0 = 2.9$.
 - (a) Write down the iteration formula
 - (b) Perform 4 iterations
 - (c) Give your answer to 5 decimal places
 - (d) Compare the convergence with bisection method
12. Apply Newton-Raphson to find $\sqrt{18}$ by solving $x^2 - 18 = 0$:
 - (a) Derive the iteration formula
 - (b) Start with $x_0 = 4$ and perform 4 iterations
 - (c) Compare with the exact value
 - (d) Explain why this converges so quickly

13. Solve $\sec x = x$ using Newton-Raphson method:
- (a) Rearrange to standard form $f(x) = 0$
 - (b) Find $f'(x)$ and write the iteration formula
 - (c) Use $x_0 = 1.2$ and perform 5 iterations
 - (d) Check your answer by substitution
14. Use Newton-Raphson to solve $e^x - 5x - 1 = 0$:
- (a) Find the iteration formula
 - (b) Starting with $x_0 = 2.5$, find the root to 6 decimal places
 - (c) Starting with $x_0 = 0.2$, find the other root
 - (d) Discuss the importance of choosing good initial values
15. Investigate the convergence of Newton-Raphson for $f(x) = x^3 - 12x + 8$:
- (a) Find all roots using different starting values
 - (b) Identify cases where the method fails to converge
 - (c) Explain why some starting values lead to divergence
 - (d) Sketch the function and its derivative to illustrate your findings
16. For the equation $x^4 - 12x^2 + 32 = 0$:
- (a) Solve exactly by substitution
 - (b) Use Newton-Raphson to find all four roots
 - (c) Compare numerical and exact solutions
 - (d) Discuss which starting values work best for each root

Section D: Fixed Point Iteration

17. Rearrange $x^2 - 7x + 5 = 0$ into the form $x = g(x)$ in different ways:
- (a) $x = \frac{x^2+5}{7}$
 - (b) $x = 7 - \frac{5}{x}$
 - (c) $x = \sqrt{7x - 5}$
 - (d) Test each rearrangement for convergence near $x = 6.193$
18. Use fixed point iteration to solve $x = \sec x$:
- (a) Use the iteration $x_{n+1} = \sec x_n$ with $x_0 = 1.2$
 - (b) Perform 10 iterations
 - (c) Plot the values to show convergence
 - (d) Explain why this method converges
19. Solve $x^3 - 4x - 4 = 0$ using fixed point iteration:
- (a) Try the rearrangement $x = x^3 - 4$
 - (b) Try the rearrangement $x = \sqrt[3]{4x + 4}$
 - (c) Try the rearrangement $x = \frac{4}{x^2 - 4}$
 - (d) Determine which rearrangements converge and why
20. For the equation $e^x = 6x$:

- (a) Show that $x = \frac{e^x}{6}$ diverges from $x_0 = 1$
 - (b) Try $x = \ln(6x)$ starting from $x_0 = 1.5$
 - (c) Explain the convergence behavior using $|g'(x)|$
 - (d) Find the root to 4 decimal places
21. Investigate the convergence condition $|g'(x)| < 1$:
- (a) For $g(x) = \frac{x^2+5}{7}$, find $g'(x)$ and determine convergence regions
 - (b) For $g(x) = \sqrt{7x-5}$, analyze convergence
 - (c) Explain why some iterations converge while others diverge
 - (d) Relate this to the graphical interpretation of fixed point iteration

Section E: Trapezium Rule

22. Use the trapezium rule with 4 strips to approximate:

- (a) $\int_0^2 x^5 dx$
- (b) $\int_1^3 \frac{1}{x^4} dx$
- (c) $\int_0^1 e^{4x} dx$
- (d) $\int_0^{\pi/2} \sin 3x dx$

Compare with exact values and calculate absolute errors.

23. Apply the trapezium rule to $\int_0^1 \sqrt{1+x^5} dx$:
- (a) Use 2 strips, then 4 strips, then 8 strips
 - (b) Comment on how the approximation improves
 - (c) Estimate the true value of the integral
 - (d) Explain why exact integration is difficult
24. For $\int_1^2 \frac{1}{x^5} dx$:
- (a) Calculate the exact value
 - (b) Use trapezium rule with $n = 2, 4, 8$ strips
 - (c) Calculate the error for each approximation
 - (d) Show that halving the strip width approximately quarters the error
25. Use the trapezium rule to estimate $\int_0^1 e^{-x^5} dx$:
- (a) Use 5 ordinates (4 strips)
 - (b) Use 9 ordinates (8 strips)
 - (c) Compare your answers and estimate the accuracy
 - (d) This integral cannot be expressed in elementary functions - explain why numerical methods are essential
26. A curve passes through points $(0, 3.2)$, $(0.5, 4.1)$, $(1, 5.3)$, $(1.5, 4.8)$, $(2, 3.9)$:
- (a) Use trapezium rule to find the area under the curve
 - (b) If the y -values represent velocity in m/s and x represents time in seconds, interpret your answer
 - (c) How could you improve the accuracy?
 - (d) Discuss the limitations when working with discrete data points

Section F: Simpson's Rule

27. Use Simpson's rule with 4 strips to approximate:

- (a) $\int_0^2 x^6 dx$
- (b) $\int_1^3 \frac{1}{x^5} dx$
- (c) $\int_0^1 e^{5x} dx$
- (d) $\int_0^\pi \sin 3x dx$

Compare with exact values and trapezium rule approximations.

28. Apply Simpson's rule to $\int_0^1 \frac{1}{1+x^5} dx$:

- (a) Use 2 strips, then 4 strips, then 8 strips
- (b) Comment on the convergence pattern
- (c) Compare convergence with trapezium rule
- (d) Explain why Simpson's rule is more accurate

29. For $\int_0^5 \sqrt{25-x^2} dx$:

- (a) Recognize this as the area of a quarter circle
- (b) Use Simpson's rule with 4 and 8 strips
- (c) Compare with the exact value $\frac{25\pi}{4}$
- (d) Calculate percentage errors

30. Use Simpson's rule to estimate $\int_1^3 x^4 \ln x dx$:

- (a) Use 4 strips
- (b) Use 8 strips
- (c) The exact value is $\frac{243 \ln 3 - 80}{5}$ - verify this and calculate errors
- (d) Discuss the convergence rate

31. A dam has the following cross-sectional areas at 4m intervals:

Distance (m)	0	4	8	12	16	20
Area (m ²)	65	95	120	110	85	55

- (a) Use Simpson's rule to estimate the volume of water held
- (b) If water pressure creates 2 tonnes force per m², estimate total force
- (c) Discuss the accuracy of your approximation
- (d) What additional data would improve the estimate?

Section G: Error Analysis and Comparison of Methods

32. For $\int_0^1 x^8 dx$:

- (a) Calculate the exact value
- (b) Use trapezium rule with $n = 2, 4, 8$ strips
- (c) Use Simpson's rule with $n = 2, 4, 8$ strips
- (d) Create a table comparing errors
- (e) Verify the theoretical error formulas

33. Analyze the errors in numerical integration:
- (a) Explain why trapezium rule has error proportional to h^2
 - (b) Explain why Simpson's rule has error proportional to h^4
 - (c) For what types of functions is each method most accurate?
 - (d) Give examples where each method might be preferred
34. Compare root-finding methods for $f(x) = x^3 - 6x - 9$:
- (a) Use bisection method (6 iterations from $[2, 3]$)
 - (b) Use Newton-Raphson (4 iterations from $x_0 = 2.9$)
 - (c) Use fixed point iteration with $x = \sqrt[3]{6x + 9}$ (6 iterations from $x_0 = 2.9$)
 - (d) Compare convergence rates and accuracy
 - (e) Discuss advantages and disadvantages of each method
35. For the equation $x = x$ in $(0, \pi)$:
- (a) Explain why bisection method works reliably
 - (b) Discuss potential problems with Newton-Raphson method
 - (c) Suggest appropriate starting values and intervals
 - (d) Find the root using your preferred method
36. Error propagation in numerical methods:
- (a) If $f(2.9) = 0.22$ and $f(3) = -0.45$, estimate the error in the root found by linear interpolation
 - (b) For Newton-Raphson, if $f'(x)$ is small near the root, how does this affect convergence?
 - (c) In numerical integration, how do rounding errors accumulate?
 - (d) Suggest strategies to minimize computational errors

Section H: Advanced Applications

37. Solve systems of equations numerically. For the system: $x^2 + y^2 = 16$, $xy = 5$
- (a) Rearrange to eliminate one variable
 - (b) Solve the resulting equation using Newton-Raphson
 - (c) Find all solutions
 - (d) Verify your answers by substitution
 - (e) Compare with algebraic solution
38. A projectile's height is given by $h(t) = 40t - 6t^2$ for $t \geq 0$.
- (a) Find when the projectile hits the ground exactly
 - (b) Use numerical methods to find when $h(t) = 30$
 - (c) Find the maximum height and when it occurs
 - (d) Use numerical integration to find the total distance traveled
 - (e) Model air resistance with $h(t) = 40t - 6t^2 - 0.18t^3$ and solve numerically
39. The equation $x^3 - 7x + e = 0$ has parameter e .
- (a) For what values of e does the equation have three real roots?
 - (b) For $e = 5$, find all roots numerically

- (c) For $e = -5$, find all roots numerically
 - (d) Investigate the behavior as e varies
 - (e) Create a bifurcation diagram showing how roots change with e
40. Population growth is modeled by $\frac{dP}{dt} = rP(1 - \frac{P}{K})$ where $P(0) = P_0$.
- (a) For $r = 0.06$, $K = 2000$, $P_0 = 120$, the solution is $P(t) = \frac{2000}{1 + \frac{47}{3}e^{-0.06t}}$
 - (b) Use Newton-Raphson to find when $P(t) = 1000$
 - (c) Find when the growth rate $\frac{dP}{dt}$ is maximum
 - (d) Use numerical integration to find the total growth in the first 40 years
 - (e) Model seasonal variation with $r(t) = 0.06 + 0.015 \sin(2\pi t)$ and solve numerically
41. Financial modeling: An investment grows according to $A(t) = Pe^{rt}$ where r varies.
- (a) If $P = 3000$ and $A(8) = 4500$, find r using Newton-Raphson
 - (b) For compound interest $A = P(1 + \frac{r}{n})^{nt}$, find r when $P = 3000$, $A = 6000$, $t = 18$, $n = 1$
 - (c) Use numerical integration to find the average value of $A(t)$ over $[0, 18]$
 - (d) Model variable interest rates and compare investment strategies

Section I: Advanced Topics and Optimization

42. Multi-variable Newton-Raphson for system: $f(x, y) = x^2 + y^2 - 8 = 0$, $g(x, y) = xy - 3.5 = 0$
- (a) Set up the Jacobian matrix
 - (b) Derive the iteration formulas
 - (c) Find a solution starting from $(2.8, 1.8)$
 - (d) Compare with single-variable approach
 - (e) Discuss convergence criteria for systems
43. Optimization using numerical methods:
- (a) Find the minimum of $f(x) = x^4 - 10x^3 + 24x^2 - 18x + 7$ using Newton-Raphson on $f'(x) = 0$
 - (b) Use numerical integration to find the area under $f(x)$ from 0 to 6
 - (c) Find the point where $f(x) = 4$ has multiple solutions
 - (d) Analyze the stability of each critical point
44. Adaptive integration methods:
- (a) Implement Richardson extrapolation for trapezium rule
 - (b) Use adaptive Simpson's rule with error control
 - (c) Compare computational efficiency
 - (d) Apply to $\int_0^1 \frac{\sec x}{x} dx$ (using $\lim_{x \rightarrow 0} \frac{\sec x}{x} = \frac{1}{x}$)
45. Boundary value problems: Solve $y'' + 5y = x$ with $y(0) = 0$, $y(\pi) = 0$:
- (a) Convert to a system of first-order equations
 - (b) Use shooting method with Newton-Raphson
 - (c) Implement finite difference method
 - (d) Compare solutions with exact answer
 - (e) Discuss numerical stability

46. Fourier analysis using numerical methods:

- (a) For $f(x) = x^4$ on $[-\pi, \pi]$, compute Fourier coefficients numerically
- (b) Use trapezium rule to evaluate $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$
- (c) Compute b_n coefficients similarly
- (d) Compare with analytical Fourier series
- (e) Discuss convergence and Gibbs phenomenon

47. Chaos and sensitivity analysis:

- (a) Study the logistic map $x_{n+1} = rx_n(1 - x_n)$
- (b) For $r = 3.5$, find the fixed point using Newton-Raphson
- (c) For $r = 3.95$, demonstrate chaotic behavior
- (d) Show sensitivity to initial conditions
- (e) Create a bifurcation diagram for $r \in [3.2, 4]$
- (f) Discuss implications for numerical accuracy

48. Monte Carlo methods:

- (a) Estimate π using random points in a unit square
- (b) Use Monte Carlo integration for $\int_0^1 e^{-x^6} dx$
- (c) Compare accuracy with deterministic methods
- (d) Analyze convergence rate ($\propto 1/\sqrt{n}$)
- (e) Discuss when Monte Carlo methods are preferred
- (f) Apply to multi-dimensional integration

49. Numerical differentiation and applications:

- (a) Derive forward, backward, and central difference formulas
- (b) Estimate $f'(3)$ for $f(x) = \cos(x^2)$ using different step sizes
- (c) Analyze truncation error vs. rounding error trade-off
- (d) Apply to find critical points of tabulated data
- (e) Use for solving differential equations numerically
- (f) Implement higher-order difference formulas

50. Spline interpolation and curve fitting:

- (a) Construct cubic spline through points $(0, 3.2)$, $(1, 3.8)$, $(2, 3.5)$, $(3, 4.1)$
- (b) Compare with polynomial interpolation
- (c) Discuss advantages of splines for numerical integration
- (d) Apply to data smoothing problems
- (e) Use for solving differential equations
- (f) Implement natural and clamped boundary conditions

51. Design a comprehensive numerical analysis project:

- (a) Choose a real-world problem requiring multiple numerical methods
- (b) Implement root-finding, integration, and optimization
- (c) Analyze error propagation and computational complexity
- (d) Compare different numerical approaches
- (e) Validate results against known solutions where possible
- (f) Present findings with appropriate visualizations
- (g) Discuss limitations and potential improvements

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 200

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