A Level Pure Mathematics
Practice Test 2: Sequences and Series

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Arithmetic Sequences

1. For the arithmetic sequence $7, 12, 17, 22, 27, \ldots$:
   (a) Find the first term a and common difference d
   (b) Find the general term u_n
   (c) Calculate u_{25}
   (d) Find which term equals 97
   (e) Determine if 150 is a term in the sequence
2. An arithmetic sequence has $u_4 = 23$ and $u_9 = 43$.
   (a) Find the first term and common difference
   (b) Write the general term u_n
   (c) Calculate u_{20}
   (d) Find the first term to exceed 100
   (e) Determine the largest value of n for which $u_n < 150$
3. The nth term of an arithmetic sequence is $u_n = 4n + 1$.
   (a) Write down the first five terms
   (b) Find the common difference
   (c) Calculate u_{40}
   (d) Find the sum of the first 30 terms
   (e) For what value of n is $u_n = 121$?
4. Three numbers $y - 2d$, y, and $y + 2d$ are in arithmetic progression with sum 36 and product 1620.
   (a) Find the value of y
   (b) Set up an equation for d
   (c) Solve to find the values of d
   (d) Write down the three numbers for each case
5. An arithmetic sequence has first term a and common difference d.

(a) If $u_m = x$ and $u_n = y$ where $m \neq n$, find u_{m+n} in terms of x, y, m, and n

(b) Show that the sequence of terms $u_1 + u_n$, $u_2 + u_{n-1}$, $u_3 + u_{n-2}$, ... is constant

(c) Prove that $S_n = \frac{n}{2}(u_1 + u_n)$

(d) If $S_p = q$ and $S_q = p$ where $p \neq q$, find S_{p+q}

Section B: Arithmetic Series

6. Calculate the sum of these arithmetic series:

   (a) $4 + 9 + 14 + 19 + \dots$ (first 18 terms)

   (b) $25 + 21 + 17 + 13 + \dots$ (first 10 terms)

   (c) $\frac{3}{4} + \frac{5}{4} + \frac{7}{4} + \frac{9}{4} + \dots$ (first 20 terms)

   (d) The series with first term 12, last term 72, and 9 terms

7. An arithmetic series has first term 6 and common difference 4.

   (a) Find the sum of the first 25 terms

   (b) Find the smallest value of n for which $S_n \geq 2000$

   (c) If the sum of the first n terms is 816, find n

   (d) Express S_n in terms of n

8. The sum of the first n terms of an arithmetic series is $S_n = 3n^2 - n$.

   (a) Find the first term u_1

   (b) Find u_2 and u_3

   (c) Determine the common difference

   (d) Find the general term u_n

   (e) Verify using the formula $u_n = S_n - S_{n-1}$ for $n \geq 2$

9. Find the sum of:

   (a) All multiples of 9 between 100 and 1000

   (b) All odd integers from 1 to 99

   (c) All integers from 1 to 200 that are divisible by 4

   (d) The integers from 1 to 150 that are divisible by 5 or 7

10. An arithmetic series has $S_{15} = 330$ and $S_{25} = 850$.

    (a) Find the first term and common difference

    (b) Calculate S_{40}

    (c) Find the 20th term

    (d) Determine when the sum first exceeds 3000

Section C: Geometric Sequences

11. For the geometric sequence $2, 8, 32, 128, 512, \dots$:

    (a) Find the first term a and common ratio r

    (b) Find the general term u_n

    (c) Calculate u_{12}

    (d) Find which term equals 2048

(e) Determine if 4096 is a term in the sequence

12. A geometric sequence has $u_3 = 12$ and $u_6 = 324$.

    (a) Find the common ratio r
    (b) Find the first term a
    (c) Write the general term u_n
    (d) Calculate u_{10}
    (e) Find the first term to exceed 50000

13. The nth term of a geometric sequence is $u_n = 3 \times 4^{n-1}$.

    (a) Write down the first five terms
    (b) Find the common ratio
    (c) Calculate u_8
    (d) Find the sum of the first 6 terms
    (e) For what value of n is $u_n = 12288$?

14. Three numbers $\frac{z}{s}$, z, and zs are in geometric progression with sum 42 and product 512.

    (a) Find the value of z
    (b) Set up an equation for s
    (c) Solve to find the values of s
    (d) Write down the three numbers for each case

15. A geometric sequence has first term a and common ratio r.

    (a) If $u_m = p$ and $u_n = q$ where $m \neq n$, find u_{m+n} in terms of p, q, m, and n
    (b) Show that $u_1 \cdot u_{2n-1} = u_2 \cdot u_{2n-2} = u_3 \cdot u_{2n-3} = \ldots = u_n^2$
    (c) Prove that if three terms u_i, u_j, u_k are in arithmetic progression, then $\frac{1}{u_i}$, $\frac{1}{u_j}$, $\frac{1}{u_k}$ are in harmonic progression
    (d) Show that $(u_n)^2 = u_{n-1} \cdot u_{n+1}$ for any $n \geq 2$

Section D: Geometric Series

16. Calculate the sum of these geometric series:

    (a) $5 + 15 + 45 + 135 + \ldots$ (first 7 terms)
    (b) $1 - 4 + 16 - 64 + \ldots$ (first 8 terms)
    (c) $\frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \frac{3}{32} + \ldots$ (first 10 terms)
    (d) $80 + 72 + 64.8 + 58.32 + \ldots$ (first 12 terms)

17. A geometric series has first term 8 and common ratio $\frac{3}{4}$.

    (a) Find the sum of the first 12 terms
    (b) Find the smallest value of n for which $S_n \geq 30$
    (c) Calculate the sum to infinity
    (d) Find how many terms are needed for the sum to be within 0.1 of the sum to infinity

18. The sum of the first n terms of a geometric series is $S_n = 4(3^n - 1)$.

    (a) Find the first term u_1

(b) Find u_2 and u_3
(c) Determine the common ratio
(d) Find the general term u_n
(e) Verify using the formula $u_n = S_n - S_{n-1}$ for $n \geq 2$

19. Evaluate these infinite geometric series:
   (a) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots$
   (b) $4 - 2 + 1 - \frac{1}{2} + \dots$
   (c) $\frac{5}{6} + \frac{5}{18} + \frac{5}{54} + \frac{5}{162} + \dots$
   (d) $0.4 + 0.04 + 0.004 + 0.0004 + \dots$

20. A geometric series has $S_4 = 30$ and $S_8 = 510$.
   (a) Set up equations for the first term and common ratio
   (b) Solve to find a and r
   (c) Calculate S_{12}
   (d) Find the sum to infinity (if it exists)
   (e) Determine the first term to exceed 2000

Section E: Sigma Notation

21. Evaluate these sums:
   (a) $\sum_{r=1}^{12}(3r+1)$
   (b) $\sum_{r=1}^{25}(4r-3)$
   (c) $\sum_{r=1}^{18} r^2$
   (d) $\sum_{r=1}^{10}(2r^2+r)$

22. Express these series using sigma notation:
   (a) $7 + 11 + 15 + 19 + \dots + 43$
   (b) $3 + 12 + 48 + 192 + \dots + 12288$
   (c) $2^3 + 3^3 + 4^3 + \dots + 12^3$
   (d) $\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \dots + \frac{1}{80}$

23. Use the standard formulae to evaluate:
   (a) $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$: Find $\sum_{r=1}^{75} r$
   (b) $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$: Find $\sum_{r=1}^{25} r^2$
   (c) $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$: Find $\sum_{r=1}^{12} r^3$
   (d) $\sum_{r=1}^{40}(3r^2 - 2r + 1)$

24. Simplify these expressions:
   (a) $\sum_{r=1}^{n}(kr+c)$ in terms of k, c, and n
   (b) $\sum_{r=1}^{n}(2r^2 - 3r + 1)$
   (c) $\sum_{r=1}^{n}(r+2)^2$
   (d) $\sum_{r=1}^{n} r(2r-1)$

25. Prove these results:

(a) $\sum_{r=1}^{n}(3r-2) = \frac{3n^2+n}{2}$
(b) $\sum_{r=1}^{n} r(r+2) = \frac{n(n+1)(n+5)}{3}$
(c) $\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$
(d) $\sum_{r=1}^{n}(r^2-(r-1)^2) = n$

Section F: Binomial Expansion - Integer Powers

26. Expand using the binomial theorem:
    (a) $(x+3)^5$
    (b) $(3x-2)^4$
    (c) $(2-x)^6$
    (d) $(2x+\frac{1}{x})^5$
27. Find the specified terms in these expansions:
    (a) The coefficient of x^4 in $(3x+1)^8$
    (b) The coefficient of x^6 in $(x-3)^9$
    (c) The constant term in $(x^3+\frac{1}{x})^8$
    (d) The coefficient of x^{-3} in $(3x-\frac{2}{x^2})^9$
28. Use the binomial theorem to evaluate:
    (a) $(1.03)^4$ to 6 decimal places
    (b) $(0.97)^5$ to 5 decimal places
    (c) $(1.05)^3$ exactly
    (d) 98^4 by writing it as $(100-2)^4$
29. In the expansion of $(1+bx)^m$:
    (a) The coefficient of x is 15 and the coefficient of x^2 is 90. Find b and m.
    (b) Find the coefficient of x^3
    (c) Write out the first four terms of the expansion
    (d) For what values of x does the expansion converge?
30. The coefficient of x^k in the expansion of $(1+x)^m$ is $\binom{m}{k}$.
    (a) Show that $\binom{m}{0}+\binom{m}{1}+\binom{m}{2}+\ldots+\binom{m}{m} = 2^m$
    (b) Prove that $\binom{m}{k} = \binom{m-1}{k-1}+\binom{m-1}{k}$
    (c) Use this identity to find $\binom{8}{3}$ from known values
    (d) Show that $\binom{m}{0}-\binom{m}{1}+\binom{m}{2}-\ldots+(-1)^m\binom{m}{m} = 0$ for $m \geq 1$

Section G: Binomial Expansion - Non-Integer Powers

31. Expand these expressions up to and including the term in x^3:
    (a) $(1+x)^{-1/2}$
    (b) $(1-x)^{-2}$
    (c) $(1+3x)^{1/3}$
    (d) $(1-2x)^{-1/2}$

32. Find the first four terms in the expansion of:
    (a) $(9+x)^{1/2}$
    (b) $(16-x)^{-1/2}$
    (c) $\frac{1}{(2+x)^2}$
    (d) $\sqrt{4-3x}$

33. State the range of values of x for which these expansions are valid:
    (a) $(1+4x)^{-1} = 1 - 4x + 16x^2 - 64x^3 + \dots$
    (b) $(1-3x)^{1/2} = 1 - \frac{3x}{2} - \frac{9x^2}{8} - \frac{9x^3}{16} - \dots$
    (c) $(3+x)^{-1} = \frac{1}{3} - \frac{x}{9} + \frac{x^2}{27} - \frac{x^3}{81} + \dots$
    (d) $\frac{1}{\sqrt{9-x}} = \frac{1}{3} + \frac{x}{54} + \frac{x^2}{1458} + \dots$

34. Use binomial expansions to find approximations:
    (a) $\sqrt{1.04}$ to 5 decimal places
    (b) $\frac{1}{\sqrt{0.96}}$ to 4 decimal places
    (c) $(1.03)^{-3}$ to 6 decimal places
    (d) $\sqrt[3]{1.06}$ to 5 decimal places

35. Find the coefficient of x^2 in the expansion of:
    (a) $(1+x)^{-1/2}(1+x)^{1/2}$
    (b) $(1+3x)^{-1}(1-x)^2$
    (c) $\frac{1-x}{\sqrt{1+x}}$
    (d) $(1-x+x^2)(1+x)^{-2}$

Section H: Mixed Series and Advanced Topics

36. A sequence is defined by $u_1 = 3$ and $u_{n+1} = 2u_n + 5$ for $n \geq 1$.
    (a) Find the first five terms
    (b) Prove by induction that $u_n = 8 \times 2^{n-1} - 5$
    (c) Calculate u_{12}
    (d) Find the sum of the first 10 terms

37. The sequence $\{w_n\}$ satisfies $w_n = 3w_{n-1} - 2w_{n-2}$ with $w_1 = 4$ and $w_2 = 7$.
    (a) Find the first six terms
    (b) Show that the characteristic equation is $r^2 - 3r + 2 = 0$
    (c) Solve to find $r = 2$ and $r = 1$
    (d) Use the general solution $w_n = A \cdot 2^n + B \cdot 1^n$ to find A and B
    (e) Write the explicit formula for w_n

38. Consider the series $\sum_{r=1}^{\infty} \frac{1}{r(r+2)}$.
    (a) Use partial fractions to show that $\frac{1}{r(r+2)} = \frac{1}{2}\left(\frac{1}{r} - \frac{1}{r+2}\right)$
    (b) Write out the first few terms and observe the telescoping pattern
    (c) Find the sum of the first n terms

(d) Determine the sum to infinity

39. The Lucas sequence is defined by $L_1 = 1$, $L_2 = 3$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 3$.

   (a) Write down the first 10 terms
   (b) Calculate the ratios $\frac{L_{n+1}}{L_n}$ for $n = 1, 2, 3, \ldots, 9$
   (c) Show that these ratios approach the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$
   (d) Prove that $L_n = \phi^n + (-\phi)^{-n}$ for all $n \geq 1$

40. A pendulum is released from rest. On each swing, it travels $\frac{4}{5}$ of the distance of the previous swing. The first swing covers 2 meters.

   (a) Find the distance covered on the 8th swing
   (b) Calculate the total distance traveled when the pendulum comes to rest
   (c) Find the number of swings needed to cover 95% of the total distance
   (d) Model the swing distances as a geometric sequence and analyze its convergence

Section I: Applications and Problem Solving

41. A mortgage of £150,000 is taken out at 6% annual compound interest. Monthly payments of £1200 are made.

   (a) Set up a recurrence relation for the amount owed after n months
   (b) Find the amount owed after 24 months
   (c) Determine how many months it takes to pay off the mortgage
   (d) Calculate the total interest paid

42. A population of rabbits grows according to the model: each pair produces 4 offspring every 2 months. Initially, there are 10 pairs.

   (a) Model the population as a geometric sequence
   (b) Find the population after 18 months
   (c) After how many months will the population exceed 100,000 pairs?
   (d) If predation reduces the population by 15% every 6 months, modify the model and find the population after 24 months

43. A fractal pattern is created using triangles with areas forming a geometric sequence: 16, 4, 1, $\frac{1}{4}$, ... cm².

   (a) Find the total area of all the triangles
   (b) If each triangle has a perimeter proportional to the square root of its area, find the total perimeter
   (c) If paint costs £5 per cm² to apply, find the total cost to paint all triangles
   (d) What fraction of the total area is occupied by the first 4 triangles?

44. A chemical reaction follows first-order kinetics. Initially, 200 mg of reactant is present. Every minute, 15% decomposes, and 5 mg of fresh reactant is added.

   (a) Set up a recurrence relation for the amount after n minutes
   (b) Find the amount present after 10 minutes
   (c) Determine the long-term equilibrium amount
   (d) After how many minutes is the amount within 2% of the equilibrium?

45. A retirement plan involves saving £2000 in the first year, £2200 in the second year, £2420 in the third year, and so on (increasing by 10% each year) for 30 years.

    (a) Model the annual savings as a geometric sequence

    (b) Find the total amount saved over 30 years

    (c) If each year's savings earns 7% annual compound interest from when it's deposited, find the total value after 30 years

    (d) Compare this with saving £2000 annually at 7% compound interest for 30 years

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

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