A Level Pure Mathematics
Practice Test 6: Sequences and Series

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Arithmetic Sequences

1. For the arithmetic sequence $15, 23, 31, 39, 47, \ldots$:
   (a) Find the first term a and common difference d
   (b) Find the general term u_n
   (c) Calculate u_{24}
   (d) Find which term equals 207
   (e) Determine if 350 is a term in the sequence
2. An arithmetic sequence has $u_8 = 47$ and $u_{14} = 77$.
   (a) Find the first term and common difference
   (b) Write the general term u_n
   (c) Calculate u_{28}
   (d) Find the first term to exceed 140
   (e) Determine the largest value of n for which $u_n < 220$
3. The nth term of an arithmetic sequence is $u_n = 8n - 6$.
   (a) Write down the first five terms
   (b) Find the common difference
   (c) Calculate u_{42}
   (d) Find the sum of the first 32 terms
   (e) For what value of n is $u_n = 218$?
4. Three numbers $t-6g$, t, and $t+6g$ are in arithmetic progression with sum 84 and product 24192.
   (a) Find the value of t
   (b) Set up an equation for g
   (c) Solve to find the values of g
   (d) Write down the three numbers for each case
5. An arithmetic sequence has first term a and common difference d.

(a) If three terms u_i, u_j, u_k satisfy $u_i + u_k = 2u_j$, prove that $i + k = 2j$

(b) Show that the reciprocals $\frac{1}{u_p}$, $\frac{1}{u_q}$, $\frac{1}{u_r}$ are in harmonic progression when u_p, u_q, u_r are in arithmetic progression

(c) Prove that if $S_m = S_n$ where $m \neq n$, then the arithmetic mean of all terms from u_1 to u_{m+n} equals the common value $\frac{S_m}{m} = \frac{S_n}{n}$

(d) If the sum of the first p terms is α times the sum of the first q terms, find a relationship between p, q, α, a, and d

Section B: Arithmetic Series

6. Calculate the sum of these arithmetic series:

   (a) $12 + 20 + 28 + 36 + \ldots$ (first 26 terms)

   (b) $45 + 40 + 35 + 30 + \ldots$ (first 20 terms)

   (c) $\frac{4}{9} + \frac{7}{9} + \frac{10}{9} + \frac{13}{9} + \ldots$ (first 32 terms)

   (d) The series with first term 24, last term 168, and 25 terms

7. An arithmetic series has first term 15 and common difference 9.

   (a) Find the sum of the first 28 terms

   (b) Find the smallest value of n for which $S_n \geq 4000$

   (c) If the sum of the first n terms is 3192, find n

   (d) Express S_n in terms of n

8. The sum of the first n terms of an arithmetic series is $S_n = 6n^2 - 4n$.

   (a) Find the first term u_1

   (b) Find u_2 and u_3

   (c) Determine the common difference

   (d) Find the general term u_n

   (e) Verify using the formula $u_n = S_n - S_{n-1}$ for $n \geq 2$

9. Find the sum of:

   (a) All multiples of 13 between 200 and 1500

   (b) All integers from 1 to 200 that are divisible by 9

   (c) All odd integers from 25 to 195

   (d) The integers from 1 to 100 that are divisible by 6 or 8

10. An arithmetic series has $S_{18} = 738$ and $S_{26} = 1378$.

    (a) Find the first term and common difference

    (b) Calculate S_{40}

    (c) Find the 22nd term

    (d) Determine when the sum first exceeds 5000

Section C: Geometric Sequences

11. For the geometric sequence $7, 28, 112, 448, 1792, \ldots$:
    (a) Find the first term a and common ratio r
    (b) Find the general term u_n
    (c) Calculate u_{14}
    (d) Find which term equals 7168
    (e) Determine if 458752 is a term in the sequence

12. A geometric sequence has $u_7 = 192$ and $u_{10} = 1536$.
    (a) Find the common ratio r
    (b) Find the first term a
    (c) Write the general term u_n
    (d) Calculate u_{15}
    (e) Find the first term to exceed 1000000

13. The nth term of a geometric sequence is $u_n = 12 \times 4^{n-1}$.
    (a) Write down the first five terms
    (b) Find the common ratio
    (c) Calculate u_{11}
    (d) Find the sum of the first 7 terms
    (e) For what value of n is $u_n = 12288$?

14. Three numbers $\frac{w}{v}$, w, and wv are in geometric progression with sum 182 and product 4096.
    (a) Find the value of w
    (b) Set up an equation for v
    (c) Solve to find the values of v
    (d) Write down the three numbers for each case

15. A geometric sequence has first term a and common ratio r.
    (a) If the sum of the first n terms is S_n and the sum of their squares is T_n, find a relationship between S_n, T_n, a, and r
    (b) Show that if three terms u_m, u_n, u_p are in arithmetic progression, then $r^{n-m} + r^{p-n} = 2$ when $m < n < p$
    (c) Prove that the sequence of ratios $\frac{u_2}{u_1}, \frac{u_3}{u_2}, \frac{u_4}{u_3}, \ldots$ is constant
    (d) If P_n represents the product of the first n terms, show that $P_n^2 = (u_1 \cdot u_n)^n$ when n is even

Section D: Geometric Series

16. Calculate the sum of these geometric series:
    (a) $13 + 39 + 117 + 351 + \ldots$ (first 11 terms)
    (b) $5 - 20 + 80 - 320 + \ldots$ (first 10 terms)
    (c) $\frac{4}{9} + \frac{4}{27} + \frac{4}{81} + \frac{4}{243} + \ldots$ (first 8 terms)
    (d) $144 + 108 + 81 + 60.75 + \ldots$ (first 18 terms)

17. A geometric series has first term 21 and common ratio $\frac{5}{7}$.

(a) Find the sum of the first 30 terms
(b) Find the smallest value of n for which $S_n \geq 48$
(c) Calculate the sum to infinity
(d) Find how many terms are needed for the sum to be within 0.01 of the sum to infinity

18. The sum of the first n terms of a geometric series is $S_n = 8(7^n - 1)$.

(a) Find the first term u_1
(b) Find u_2 and u_3
(c) Determine the common ratio
(d) Find the general term u_n
(e) Verify using the formula $u_n = S_n - S_{n-1}$ for $n \geq 2$

19. Evaluate these infinite geometric series:

(a) $1 + \frac{4}{9} + \frac{16}{81} + \frac{64}{729} + \ldots$
(b) $15 - 7.5 + 3.75 - 1.875 + \ldots$
(c) $\frac{11}{12} + \frac{11}{60} + \frac{11}{300} + \frac{11}{1500} + \ldots$
(d) $0.8 + 0.08 + 0.008 + 0.0008 + \ldots$

20. A geometric series has $S_8 = 510$ and $S_{16} = 65790$.

(a) Set up equations for the first term and common ratio
(b) Solve to find a and r
(c) Calculate S_{24}
(d) Find the sum to infinity (if it exists)
(e) Determine the first term to exceed 100000

Section E: Sigma Notation

21. Evaluate these sums:

(a) $\sum_{r=1}^{22}(7r + 5)$
(b) $\sum_{r=1}^{45}(8r - 7)$
(c) $\sum_{r=1}^{36} r^2$
(d) $\sum_{r=1}^{20}(6r^2 + 4r)$

22. Express these series using sigma notation:

(a) $14 + 22 + 30 + 38 + \ldots + 78$
(b) $10 + 50 + 250 + 1250 + \ldots + 31250$
(c) $4^3 + 6^3 + 8^3 + 10^3 + \ldots + 22^3$
(d) $\frac{1}{7} + \frac{1}{18} + \frac{1}{33} + \frac{1}{52} + \ldots + \frac{1}{138}$

23. Use the standard formulae to evaluate:

(a) $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$: Find $\sum_{r=1}^{105} r$
(b) $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$: Find $\sum_{r=1}^{45} r^2$
(c) $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$: Find $\sum_{r=1}^{30} r^3$
(d) $\sum_{r=1}^{55}(7r^2 - 6r + 5)$

24. Simplify these expressions:
    (a) $\sum_{r=1}^{n}(wr+z)$ in terms of w, z, and n
    (b) $\sum_{r=1}^{n}(6r^2-4r+2)$
    (c) $\sum_{r=1}^{n}(5r+3)^2$
    (d) $\sum_{r=1}^{n} r(6r+4)$
25. Prove these results:
    (a) $\sum_{r=1}^{n}(7r-6)=\frac{n(7n-5)}{2}$
    (b) $\sum_{r=1}^{n} r(r+6)=\frac{n(n+1)(n+17)}{3}$
    (c) $\sum_{r=1}^{n}\frac{1}{(6r-5)(6r+1)}=\frac{n}{6n+1}$
    (d) $\sum_{r=1}^{n}((r+4)^2-(r+3)^2)=n(n+9)$

Section F: Binomial Expansion - Integer Powers

26. Expand using the binomial theorem:
    (a) $(x+7)^7$
    (b) $(7x-6)^5$
    (c) $(6-5x)^4$
    (d) $(6x+\frac{5}{x})^7$
27. Find the specified terms in these expansions:
    (a) The coefficient of x^8 in $(7x+5)^{12}$
    (b) The coefficient of x^{10} in $(5x-4)^{13}$
    (c) The constant term in $(x^7+\frac{6}{x^5})^4$
    (d) The coefficient of x^{-4} in $(7x^5-\frac{4}{x^3})^9$
28. Use the binomial theorem to evaluate:
    (a) $(1.07)^8$ to 6 decimal places
    (b) $(0.92)^7$ to 5 decimal places
    (c) $(1.04)^{10}$ exactly
    (d) 99^6 by writing it as $(100-1)^6$
29. In the expansion of $(1+gx)^h$:
    (a) The coefficient of x is 27 and the coefficient of x^2 is 351. Find g and h.
    (b) Find the coefficient of x^3
    (c) Write out the first four terms of the expansion
    (d) For what values of x does the expansion converge?
30. The coefficient of x^j in the expansion of $(1+x)^h$ is $\binom{h}{j}$.
    (a) Show that $\sum_{j=0}^{h}\binom{h}{j}\cdot 5^j=6^h$
    (b) Prove that $\binom{h}{1}+2\binom{h}{2}+3\binom{h}{3}+\ldots+h\binom{h}{h}=h\cdot 2^{h-1}$
    (c) Use the hockey stick identity to show that $\binom{r}{r}+\binom{r+1}{r}+\binom{r+2}{r}+\ldots+\binom{n}{r}=\binom{n+1}{r+1}$
    (d) Show that $\sum_{j=0}^{h}(-1)^j\binom{h}{j}\cdot j^3=0$ for $h\geq 3$

Section G: Binomial Expansion - Non-Integer Powers

31. Expand these expressions up to and including the term in x^3:
    (a) $(1+x)^{5/6}$
    (b) $(1-x)^{-6}$
    (c) $(1+8x)^{3/4}$
    (d) $(1-9x)^{-3/4}$

32. Find the first four terms in the expansion of:
    (a) $(64+x)^{1/2}$
    (b) $(49-x)^{-1/2}$
    (c) $\frac{1}{(6+x)^6}$
    (d) $\sqrt{36-7x}$

33. State the range of values of x for which these expansions are valid:
    (a) $(1+8x)^{-1} = 1 - 8x + 64x^2 - 512x^3 + \dots$
    (b) $(1-7x)^{1/2} = 1 - \frac{7x}{2} - \frac{49x^2}{8} - \frac{343x^3}{16} - \dots$
    (c) $(9+x)^{-1} = \frac{1}{9} - \frac{x}{81} + \frac{x^2}{729} - \frac{x^3}{6561} + \dots$
    (d) $\frac{1}{\sqrt{49-x}} = \frac{1}{7} + \frac{x}{686} + \frac{3x^2}{33614} + \dots$

34. Use binomial expansions to find approximations:
    (a) $\sqrt{1.12}$ to 5 decimal places
    (b) $\frac{1}{\sqrt{0.80}}$ to 4 decimal places
    (c) $(1.07)^{-7}$ to 6 decimal places
    (d) $\sqrt[3]{1.18}$ to 5 decimal places

35. Find the coefficient of x^2 in the expansion of:
    (a) $(1+x)^{5/6}(1-x)^{1/6}$
    (b) $(1+7x)^{-1}(1+5x)^2$
    (c) $\frac{1+5x}{\sqrt{1-x}}$
    (d) $(1+4x-3x^2)(1+x)^{-6}$

Section H: Mixed Series and Advanced Topics

36. A sequence is defined by $u_1 = 7$ and $u_{n+1} = 6u_n - 17$ for $n \geq 1$.
    (a) Find the first five terms
    (b) Prove by induction that $u_n = \frac{13\times 6^{n-1}+17}{5}$
    (c) Calculate u_{22}
    (d) Find the sum of the first 20 terms

37. The sequence $\{a_n\}$ satisfies $a_n = 7a_{n-1} - 10a_{n-2}$ with $a_1 = 5$ and $a_2 = 15$.
    (a) Find the first six terms
    (b) Show that the characteristic equation is $r^2 - 7r + 10 = 0$
    (c) Solve to find $r = 5$ and $r = 2$

(d) Use the general solution $a_n = A \cdot 5^n + B \cdot 2^n$ to find A and B
(e) Write the explicit formula for a_n

38. Consider the series $\sum_{r=1}^{\infty} \frac{4}{r(r+6)}$.

(a) Use partial fractions to show that $\frac{4}{r(r+6)} = \frac{2}{3}\left(\frac{1}{r} - \frac{1}{r+6}\right)$
(b) Write out the first few terms and observe the telescoping pattern
(c) Find the sum of the first n terms
(d) Determine the sum to infinity

39. The Perrin sequence is defined by $P_1 = 3$, $P_2 = 0$, $P_3 = 2$, and $P_n = P_{n-2} + P_{n-3}$ for $n \geq 4$.

(a) Write down the first 18 terms
(b) Calculate the ratios $\frac{P_{n+1}}{P_n}$ for several consecutive terms
(c) Investigate the convergence behavior of these ratios
(d) Study the characteristic equation $x^3 - x - 1 = 0$ and its dominant root

40. A swinging lamp gradually loses energy. Each swing covers $\frac{4}{5}$ of the distance of the previous swing. The first swing covers 15 cm.

(a) Find the distance covered on the 25th swing
(b) Calculate the total distance traveled when the lamp comes to rest
(c) Find the number of swings needed to cover 98% of the total distance
(d) If the period of each swing is proportional to the square root of its amplitude, find the total time relative to the first swing

Section I: Applications and Problem Solving

41. A home mortgage of £200,000 is taken out at 5.5% annual compound interest. Monthly payments of £1400 are made.

(a) Set up a recurrence relation for the amount owed after n months
(b) Find the amount owed after 36 months
(c) Determine how many months it takes to pay off the mortgage
(d) Calculate the total amount paid and the interest over the life of the loan

42. A computer virus spreads through a network. Each infected computer infects 6 others every 3 hours. Initially, 2 computers are infected.

(a) Model the number of infected computers as a geometric sequence
(b) Find the number of infected computers after 30 hours
(c) After how many hours will more than 50,000 computers be infected?
(d) If antivirus software reduces the infection rate to 2.5 after 18 hours, find the total infected after 48 hours

43. A fractal tree grows with branches having lengths forming the sequence: 162, 54, 18, 6, 2, $\frac{2}{3}$, ... cm.

(a) Find the total length of all the branches
(b) If each branch has cross-sectional area proportional to its length with constant 0.1, find the total cross-sectional area
(c) If pruning costs £2 per cm of branch length, find the total pruning cost

(d) What percentage of the total length is contributed by the first 4 generations?

44. A filtration system processes contaminated water. Initially, there are 900 units of contamination. Every cycle, 18% is removed, and 60 units are added from new input.

    (a) Set up a recurrence relation for the contamination level after n cycles
    (b) Find the contamination level after 25 cycles
    (c) Determine the long-term equilibrium contamination level
    (d) After how many cycles is the contamination within 1% of the equilibrium?

45. An investment scheme involves investing £6000 in the first year, £6600 in the second year, £7260 in the third year, and so on (increasing by 10% each year) for 25 years.

    (a) Model the annual investments as a geometric sequence
    (b) Find the total amount invested over 25 years
    (c) If each investment earns 9% annual compound interest from when it's made, find the total value after 25 years
    (d) Compare this with investing £6000 annually at 9% compound interest for 25 years

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

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