A Level Pure Mathematics
Practice Test 3: Differential Equations

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Introduction and Theory

1. Define and distinguish between:
   (a) Linear vs. nonlinear differential equations
   (b) Homogeneous vs. non-homogeneous equations
   (c) Complete vs. general solutions
   (d) Initial value vs. boundary value problems
   (e) Stable vs. unstable equilibrium points
   (f) Phase portraits and trajectories

2. Classify by order, degree, and type:
   (a) $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = x$
   (b) $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} - y = \sin x$
   (c) $\left(\frac{dy}{dx}\right)^2 + xy = e^x$
   (d) $\cos\left(\frac{dy}{dx}\right) = x + y$
   (e) $x^3\frac{d^3y}{dx^3} - x\frac{dy}{dx} + 2y = 0$
   (f) $\frac{dy}{dx} + xy^2 = x^3$ (Riccati equation)

3. Verify these solution pairs:
   (a) $y = \frac{C}{x^2}$ satisfies $x\frac{dy}{dx} + 2y = 0$
   (b) $y = Ce^{-x^2/2}$ satisfies $\frac{dy}{dx} + xy = 0$
   (c) $y = C_1\cos(2x) + C_2\sin(2x)$ satisfies $\frac{d^2y}{dx^2} + 4y = 0$
   (d) $y = e^{2x}(C_1 + C_2x)$ satisfies $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$

4. Derive differential equations for:
   (a) $y = Ce^{5x}$ (exponential family)
   (b) $y = C_1\sinh x + C_2\cosh x$ (hyperbolic functions)
   (c) $y^2 = 4ax$ (parabolas with vertex at origin)

(d) $xy = C$ (rectangular hyperbolas)

5. Analyze equilibrium and stability:
   (a) Find equilibria for $\frac{dy}{dx} = y(y-1)(y-2)$
   (b) Determine stability of each equilibrium point
   (c) Sketch the direction field and solution curves
   (d) Describe long-term behavior from different initial conditions

Section B: Integration Techniques

6. Solve by direct integration:
   (a) $\frac{dy}{dx} = 5x^4 - 3x^2 + 1$
   (b) $\frac{dy}{dx} = e^{-4x} + 2$
   (c) $\frac{dy}{dx} = \frac{4}{2x-3}$
   (d) $\frac{dy}{dx} = \sec^2(3x)$
   (e) $\frac{dy}{dx} = \frac{3x^2}{x^3+8}$
   (f) $\frac{dy}{dx} = x^2 e^{x^3}$

7. Find solutions with specified conditions:
   (a) $\frac{dy}{dx} = 12x^3 - 6x$, $y(2) = 10$
   (b) $\frac{dy}{dx} = 4e^{-2x}$, $y(0) = 3$
   (c) $\frac{dy}{dx} = \sin(2x)$, $y(\pi/4) = 5$
   (d) $\frac{dy}{dx} = \frac{4}{x-1}$, $y(2) = \ln 3$ (for $x > 1$)
   (e) $\frac{dy}{dx} = 2x\sqrt{x^2-4}$, $y(3) = 0$

8. Second-order problems:
   (a) $\frac{d^2y}{dx^2} = 6x^2 + 4$, $y(0) = 3$, $y'(0) = -2$
   (b) $\frac{d^2y}{dx^2} = e^{2x}$, $y(0) = 1$, $y'(0) = 2$
   (c) $\frac{d^3y}{dx^3} = 24x$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = 3$
   (d) $\frac{d^2y}{dx^2} = -\sin x$, $y(0) = 2$, $y(\pi/2) = 1$

9. Motion and physics problems:
   (a) A particle's acceleration is $a = 8t - 12$. Find $v(t)$ and $s(t)$ if $v(0) = 5$, $s(0) = -2$.
   (b) An object is thrown upward with initial velocity 25 m/s from height 30m. Find its trajectory.
   (c) A spring system has $\frac{d^2x}{dt^2} = -25x$. Find $x(t)$ if $x(0) = 3$, $\dot{x}(0) = -5$.
   (d) Find the curve whose second derivative is 6 and passes through $(0, 1)$ with slope 2.

10. Applied rate problems:
   (a) Oil leaks from a tank at rate $\frac{dV}{dt} = -0.5t^2$ L/min. Find volume lost in 4 hours.
   (b) Bacteria multiply at rate $\frac{dN}{dt} = 0.3N$. If $N(0) = 1000$, find $N(t)$.
   (c) Snow melts at rate $\frac{dh}{dt} = -k\sqrt{h}$ where $k > 0$. Solve for $h(t)$.
   (d) Temperature rises at rate $\frac{dT}{dt} = 2\sin(t/3)$ °C/hour. Find temperature change over 6 hours.

Section C: Variable Separation Methods

11. Solve these separable equations:
    (a) $\frac{dy}{dx} = 4xy^3$
    (b) $\frac{dy}{dx} = \frac{y^2}{x}$
    (c) $\frac{dy}{dx} = e^{2x-y}$
    (d) $\frac{dy}{dx} = \frac{x \sin x}{y^2}$
    (e) $\frac{dy}{dx} = \frac{\tan x}{\sec y}$
    (f) $\frac{dy}{dx} = \frac{x^3 y}{x^4+16}$
12. Particular solutions:
    (a) $\frac{dy}{dx} = 5xy$, $y(0) = 3$
    (b) $\frac{dy}{dx} = \frac{4y}{x}$, $y(1) = 6$ (for $x > 0$)
    (c) $\frac{dy}{dx} = \frac{x^2}{y^3}$, $y(0) = 4$
    (d) $\frac{dy}{dx} = y(4-y)$, $y(0) = 1$
    (e) $\frac{dy}{dx} = \frac{2x}{\sqrt{9-y^2}}$, $y(0) = 0$
13. Advanced separable forms:
    (a) $(4+y^2)\frac{dy}{dx} = 3xy$
    (b) $\frac{dy}{dx} = \frac{ye^{3x}}{x^3+1}$
    (c) $\sin^2 y \frac{dy}{dx} = \cos x$
    (d) $\frac{dy}{dx} = \frac{x^2(4+y^2)}{y(1+x^3)}$
    (e) $e^y \frac{dy}{dx} = x^2$
14. Real-world applications:
    (a) Exponential growth: $\frac{dP}{dt} = 0.025P$, $P(0) = 800$. Find doubling time.
    (b) Carbon dating: $\frac{dN}{dt} = -N$ with half-life 5730 years. Find age of artifact with 60% carbon remaining.
    (c) Cooling law: $\frac{dT}{dt} = -0.1(T-18)$. Coffee cools from 85°C to 65°C in 4 minutes. Find temperature after 10 minutes.
    (d) Logistic growth: $\frac{dP}{dt} = rP(1-\frac{P}{K})$ with $r = 0.1$, $K = 1000$, $P(0) = 50$.
15. Separability analysis:
    (a) $\frac{dy}{dx} = xy + y$ (separable)
    (b) $\frac{dy}{dx} = x + y$ (not separable)
    (c) $\frac{dy}{dx} = \sin(xy)$ (not separable)
    (d) $\frac{dy}{dx} = e^{x-2y}$ (separable)
    (e) $\frac{dy}{dx} = \frac{x^2y^2}{1+x^3}$ (separable)

Section D: First-Order Linear Equations

16. Solve using integrating factor method:
    (a) $\frac{dy}{dx} + 5y = e^{4x}$
    (b) $\frac{dy}{dx} - 3y = 2x^2$
    (c) $\frac{dy}{dx} + \frac{4y}{x} = x^2$ (for $x > 0$)
    (d) $\frac{dy}{dx} + y\cos x = \sin x \cos x$
    (e) $x\frac{dy}{dx} + 4y = x^2$
    (f) $\frac{dy}{dx} + 2xy = 3xe^{-x^2}$
17. Initial value problems:
    (a) $\frac{dy}{dx} + 4y = 8e^{2x}$, $y(0) = 3$
    (b) $\frac{dy}{dx} - 2y = 6x$, $y(0) = 2$
    (c) $\frac{dy}{dx} + 3y = 9$, $y(0) = 1$
    (d) $\frac{dy}{dx} + \frac{2y}{x} = 4x$, $y(1) = 5$ (for $x > 0$)
18. Advanced linear equations:
    (a) $\frac{dy}{dx} + y\sec x = \tan x \sec x$
    (b) $(x^2+1)\frac{dy}{dx} + 2xy = x^2 + 1$
    (c) $\frac{dy}{dx} + \frac{3y}{x^2+1} = \frac{3x}{x^2+1}$
    (d) $x^3\frac{dy}{dx} + 2x^2y = x^5$ (for $x > 0$)
19. Practical applications:
    (a) RC circuit: $RC\frac{dq}{dt} + q = CV_0$ with constant voltage. Find charge $q(t)$.
    (b) Tank mixing: 200L tank, brine enters at 3 L/min (2 kg salt/L), mixture exits at 3 L/min. Find salt content.
    (c) Savings account: $\frac{dA}{dt} = 0.06A - 1200$ (6% interest, £1200 annual withdrawal).
    (d) Terminal velocity: $m\frac{dv}{dt} + bv = mg$ for falling object with drag.
20. Method verification:
    (a) Solve $\frac{dy}{dx} = 4xy + 4x$ by separation
    (b) Solve same equation as linear: $\frac{dy}{dx} - 4xy = 4x$
    (c) Show solutions are equivalent
    (d) Compare computational efficiency of each method

Section E: Homogeneous Second-Order Equations

21. Auxiliary equation method:
    (a) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0$
    (b) $\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 25y = 0$
    (c) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0$
    (d) $\frac{d^2y}{dx^2} + 49y = 0$
    (e) $\frac{d^2y}{dx^2} - 36y = 0$

(f) $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 17y = 0$

22. Root classification and solutions:
    (a) $m^2 - 8m + 15 = 0$ (distinct real roots)
    (b) $m^2 + 12m + 36 = 0$ (repeated real root)
    (c) $m^2 + 4m + 13 = 0$ (complex conjugate roots)
    (d) $m^2 - 64 = 0$ (distinct real roots)
    (e) $m^2 + 9 = 0$ (pure imaginary roots)

23. Initial value problems:
    (a) $\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 20y = 0$, $y(0) = 2$, $y'(0) = 3$
    (b) $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0$, $y(0) = 1$, $y'(0) = -2$
    (c) $\frac{d^2y}{dx^2} + 25y = 0$, $y(0) = 0$, $y'(0) = 5$
    (d) $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 17y = 0$, $y(0) = 1$, $y'(0) = 2$

24. Solution behavior analysis:
    (a) Exponential growth/decay for real distinct roots
    (b) Oscillatory motion for complex roots
    (c) Critical damping for repeated roots
    (d) Phase relationships and amplitude modulation

25. Higher-order constant coefficient:
    (a) $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y = 0$
    (b) $\frac{d^4y}{dx^4} - 625y = 0$
    (c) General solution structure for nth order equations

Section F: Non-homogeneous Second-Order Equations

26. Undetermined coefficients:
    (a) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 18$
    (b) $\frac{d^2y}{dx^2} + 25y = 75x^2$
    (c) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 12y = e^{4x}$
    (d) $\frac{d^2y}{dx^2} + 16y = \sin(3x)$
    (e) $\frac{d^2y}{dx^2} - 16y = 4e^{4x}$
    (f) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = x^2 + 3$

27. Resonance situations:
    (a) $\frac{d^2y}{dx^2} + 25y = \cos(5x)$ (resonance)
    (b) $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = e^{3x}$ (resonance)
    (c) $\frac{d^2y}{dx^2} + 4y = \sin(2x)$ (resonance)
    (d) Explain the x multiplication rule for resonance

28. Complete IVP solutions:

(a) $\frac{d^2y}{dx^2} + 9y = 18$, $y(0) = 2$, $y'(0) = 0$
(b) $\frac{d^2y}{dx^2} - 9y = 18x$, $y(0) = 0$, $y'(0) = 3$
(c) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 4e^{-2x}$, $y(0) = 1$, $y'(0) = -1$

29. Particular integral patterns:
    (a) Polynomial forcing: trial solutions
    (b) Exponential forcing: when to include x factors
    (c) Trigonometric forcing: sine and cosine combinations
    (d) Mixed forcing: products and sums

30. Alternative methods:
    (a) Variation of parameters for $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x}$
    (b) Green's function approach concept
    (c) Operator methods introduction

Section G: Physical Applications

31. Oscillatory systems:
    (a) Mass-spring: $m\frac{d^2x}{dt^2} + kx = 0$ with $x(0) = 4$, $\dot{x}(0) = 0$, $m = 3$ kg, $k = 27$ N/m
    (b) Find natural frequency, period, and energy
    (c) Pendulum: $\frac{d^2}{dt^2} + \frac{g}{L} = 0$ for small angles
    (d) Torsional oscillations: $I\frac{d^2}{dt^2} + c = 0$

32. Damped systems:
    (a) $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$ with $m = 2$, $b = 8$, $k = 8$ (overdamped)
    (b) Critical damping: $m = 2$, $b = 8$, $k = 8$ with $x(0) = 3$, $\dot{x}(0) = -4$
    (c) Underdamped: $m = 1$, $b = 4$, $k = 13$ with $x(0) = 2$, $\dot{x}(0) = 0$
    (d) Quality factor and logarithmic decrement

33. Driven oscillations:
    (a) $\frac{d^2x}{dt^2} + 36x = 72\cos(5t)$ with zero initial conditions
    (b) Steady-state response and transient behavior
    (c) Resonance: $\frac{d^2x}{dt^2} + 25x = 50\cos(5t)$
    (d) Frequency response and amplitude curves

34. Electrical systems:
    (a) RLC circuit: $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V(t)$
    (b) Parameters: $L = 0.5$ H, $R = 6$, $C = 0.2$ F, $V = 15$ V
    (c) Natural frequency and damping characteristics
    (d) AC response with $V(t) = V_0\cos(t)$

35. Population and economic dynamics:
    (a) Second-order population model: $\frac{d^2P}{dt^2} + \frac{dP}{dt} + P = K$
    (b) Economic cycles: $\frac{d^2Y}{dt^2} + a\frac{dY}{dt} + bY = G$ (income dynamics)
    (c) Stability and equilibrium analysis
    (d) Phase space interpretation

Section H: Special Methods and Advanced Topics

36. Homogeneous equations (first-order):
    (a) $\frac{dy}{dx} = \frac{3x+2y}{x}$ (substitution $v = \frac{y}{x}$)
    (b) $\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$
    (c) $(2x^2 + 3xy)dx + (x^2 + 2xy)dy = 0$
    (d) Test for homogeneity: degree verification
37. Bernoulli equations:
    (a) $\frac{dy}{dx} + 4y = 2xy^3$ (substitution $v = y^{1-n}$)
    (b) $x\frac{dy}{dx} + 3y = y^4$
    (c) $\frac{dy}{dx} - \frac{3y}{x} = \frac{y^2}{x^3}$
38. Exact equations:
    (a) $(4x^3 + 3y)dx + (3x + 2y)dy = 0$ (test: $\frac{M}{y} = \frac{N}{x}$)
    (b) $(e^x \cos y + 2x)dx + (2y - e^x \sin y)dy = 0$
    (c) Integrating factors when not exact
39. Reduction techniques:
    (a) $\frac{d^2y}{dx^2} + \frac{3}{x}\frac{dy}{dx} = 0$ (substitute $v = \frac{dy}{dx}$)
    (b) $y\frac{d^2y}{dx^2} = 3\left(\frac{dy}{dx}\right)^2$
    (c) Euler equations: $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = 0$
40. Systems of equations:
    (a) $\frac{dx}{dt} = 4x + 2y$, $\frac{dy}{dt} = 2x + 4y$
    (b) Matrix eigenvalue method
    (c) Phase plane analysis
    (d) Stability classification

Section I: Comprehensive Modeling Project

41. Select one major application and complete full analysis:
    (a) Epidemic spread: SEIR model with demographics
    (b) Predator-prey with environmental factors
    (c) Chemical reactor with multiple reactions
    (d) Economic growth with technological change
    (e) Climate dynamics with feedback loops
    (f) Structural vibrations in engineering

    For your chosen project, provide:
    (a) Mathematical derivation from physical principles
    (b) Classification and solution strategy
    (c) Analytical solution where possible
    (d) Numerical methods if needed

   (e) Parameter estimation and validation
   (f) Sensitivity analysis and predictions
   (g) Model limitations and improvements
   (h) Graphical presentation and interpretation

42. Numerical analysis:
   (a) Euler's method for $\frac{dy}{dx} = x^2 + y$, $y(0) = 1$
   (b) Improved Euler (Heun's method)
   (c) Fourth-order Runge-Kutta method
   (d) Error analysis and step size selection

43. Boundary value problems:
   (a) $\frac{d^2y}{dx^2} + \lambda^2 y = 0$ with $y(0) = y(L) = 0$
   (b) Eigenvalue problems and eigenfunctions
   (c) Sturm-Liouville theory introduction
   (d) Applications to heat and wave equations

44. Advanced theory:
   (a) Existence and uniqueness theorems
   (b) Picard iteration method
   (c) Lipschitz conditions
   (d) Comparison of analytical vs. numerical approaches

45. Review and synthesis:
   (a) Master classification scheme
   (b) Solution method selection flowchart
   (c) Common error patterns and prevention
   (d) Historical development and modern applications
   (e) Connections to other mathematical areas

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 250

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