# A Level Pure Mathematics
# Practice Test 1: Integration

**Instructions:**
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

## Section A: Basic Integration - Polynomials

1. Find these indefinite integrals:
   (a) $\int (3x^2 + 4x - 5)\,dx$
   (b) $\int (2x^3 - x^2 + 6x + 1)\,dx$
   (c) $\int (x^4 - 2x + 3)\,dx$
   (d) $\int (5x^2 + \frac{1}{2}x - 7)\,dx$
   (e) $\int (x-1)^2\,dx$
   (f) $\int (2x+3)(x-1)\,dx$

2. Integrate these functions involving negative and fractional powers:
   (a) $\int x^{-2}\,dx$
   (b) $\int (3x^{-1} + 2x^{\frac{1}{2}})\,dx$
   (c) $\int \frac{1}{x^3}\,dx$
   (d) $\int \sqrt{x}\,dx$
   (e) $\int \frac{2}{\sqrt{x}}\,dx$
   (f) $\int (x^{\frac{3}{2}} - x^{-\frac{1}{2}})\,dx$

3. Find these integrals by expanding first:
   (a) $\int \frac{x^3+2x^2-x}{x}\,dx$
   (b) $\int \frac{x^2-4}{x}\,dx$
   (c) $\int \frac{(x+1)^2}{x}\,dx$
   (d) $\int \frac{x^3-8}{x^2}\,dx$

4. Evaluate these definite integrals:
   (a) $\int_0^2 (x^2 + 3x + 1)\,dx$
   (b) $\int_1^3 (2x - 1)\,dx$
   (c) $\int_{-1}^1 x^3\,dx$
   (d) $\int_0^4 \sqrt{x}\,dx$

5. Find the function $f(x)$ given:
   (a) $f'(x) = 3x^2 - 2x + 1$ and $f(0) = 5$
   (b) $f'(x) = 6x - 4$ and $f(1) = 3$
   (c) $f''(x) = 12x + 2$, $f'(0) = 1$, and $f(0) = 0$
   (d) $f'(x) = \frac{1}{x^2}$ for $x > 0$ and $f(1) = 2$

## Section B: Integration of Standard Functions

6. Integrate these exponential and logarithmic functions:
   (a) $\int e^x \, dx$
   (b) $\int 3e^x \, dx$
   (c) $\int e^{2x} \, dx$
   (d) $\int e^{-x} \, dx$
   (e) $\int \frac{1}{x} \, dx$ for $x > 0$
   (f) $\int \frac{2}{x} \, dx$

7. Integrate these trigonometric functions:
   (a) $\int \sin x \, dx$
   (b) $\int \cos x \, dx$
   (c) $\int 2 \sin x \, dx$
   (d) $\int 3 \cos x \, dx$
   (e) $\int \sec^2 x \, dx$
   (f) $\int \text{cosec}^2 x \, dx$

8. Find these integrals:
   (a) $\int (\sin x + \cos x) \, dx$
   (b) $\int (e^x + x^2) \, dx$
   (c) $\int (2e^x - 3 \sin x) \, dx$
   (d) $\int \left(\frac{1}{x} + x\right) \, dx$ for $x > 0$
   (e) $\int (3 \cos x + e^{-x}) \, dx$
   (f) $\int \left(x^2 + \frac{1}{x^2}\right) \, dx$ for $x > 0$

9. Evaluate these definite integrals:
   (a) $\int_0^{\pi} \sin x \, dx$
   (b) $\int_0^{\frac{\pi}{2}} \cos x \, dx$
   (c) $\int_0^1 e^x \, dx$
   (d) $\int_1^e \frac{1}{x} \, dx$
   (e) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 x \, dx$
   (f) $\int_0^{\ln 2} e^{-x} \, dx$

10. Find the exact values:
   (a) $\int_0^{\frac{\pi}{6}} 2 \sin x \, dx$
   (b) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos x \, dx$
   (c) $\int_0^{\ln 3} 2e^x \, dx$
   (d) $\int_1^{\sqrt{e}} \frac{2}{x} \, dx$

## Section C: Integration by Substitution

11. Use substitution to find these integrals:
    (a) $\int (2x+1)^3\,dx$
    (b) $\int (3x-2)^4\,dx$
    (c) $\int x(x^2+1)^2\,dx$
    (d) $\int x\sqrt{x^2+4}\,dx$
    (e) $\int \frac{x}{x^2+1}\,dx$
    (f) $\int \frac{2x}{(x^2+3)^2}\,dx$

12. Find these integrals using appropriate substitutions:
    (a) $\int \sin(2x+1)\,dx$
    (b) $\int \cos(3x-\frac{\pi}{4})\,dx$
    (c) $\int e^{2x+1}\,dx$
    (d) $\int e^{-3x}\,dx$
    (e) $\int \frac{1}{2x+5}\,dx$
    (f) $\int \frac{3}{4x-1}\,dx$

13. Use substitution for these more complex integrals:
    (a) $\int x^2(x^3+2)^4\,dx$
    (b) $\int \frac{x^2}{\sqrt{x^3+1}}\,dx$
    (c) $\int xe^{x^2}\,dx$
    (d) $\int \frac{\ln x}{x}\,dx$
    (e) $\int \sin x\cos x\,dx$
    (f) $\int \tan x\,dx$

14. Evaluate these definite integrals using substitution:
    (a) $\int_0^1 x(x^2+1)^2\,dx$
    (b) $\int_0^{\frac{\pi}{2}} \sin x\cos x\,dx$
    (c) $\int_1^2 \frac{x}{x^2+1}\,dx$
    (d) $\int_0^1 xe^{x^2}\,dx$

15. Find these integrals by recognizing the derivative pattern:
    (a) $\int \frac{2x+3}{x^2+3x+1}\,dx$
    (b) $\int \frac{3x^2-2}{x^3-2x+5}\,dx$
    (c) $\int \frac{e^x}{e^x+1}\,dx$
    (d) $\int \frac{\cos x}{\sin x}\,dx$

## Section D: Integration by Parts

16. Use integration by parts to find:
    (a) $\int xe^x \, dx$
    (b) $\int x \sin x \, dx$
    (c) $\int x \cos x \, dx$
    (d) $\int x^2 e^x \, dx$
    (e) $\int x \ln x \, dx$
    (f) $\int e^x \sin x \, dx$

17. Apply integration by parts to:
    (a) $\int \ln x \, dx$
    (b) $\int x \ln x \, dx$
    (c) $\int x^2 \ln x \, dx$
    (d) $\int \ln(x+1) \, dx$
    (e) $\int x \tan^{-1} x \, dx$
    (f) $\int x^2 \sin x \, dx$

18. Find these integrals that may require multiple applications:
    (a) $\int x^2 e^{-x} \, dx$
    (b) $\int x^2 \cos x \, dx$
    (c) $\int e^x \cos x \, dx$
    (d) $\int e^x \sin x \, dx$
    (e) $\int \sin(\ln x) \, dx$
    (f) $\int x^3 e^x \, dx$

19. Evaluate these definite integrals:
    (a) $\int_0^1 xe^x \, dx$
    (b) $\int_0^{\frac{\pi}{2}} x \sin x \, dx$
    (c) $\int_1^e x \ln x \, dx$
    (d) $\int_0^{\pi} x \cos x \, dx$

20. Prove these reduction formulas using integration by parts:
    (a) $I_n = \int x^n e^x \, dx = x^n e^x - nI_{n-1}$
    (b) $I_n = \int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2}$
    (c) Use the first formula to find $\int x^3 e^x \, dx$

## Section E: Area Under Curves

21. Find the area under these curves:
    (a) $y = x^2$ from $x = 0$ to $x = 3$
    (b) $y = 2x + 1$ from $x = 1$ to $x = 4$
    (c) $y = x^3 - x$ from $x = 0$ to $x = 2$
    (d) $y = \sin x$ from $x = 0$ to $x = \pi$

22. Calculate the area between the curve and the x-axis:
    (a) $y = x^2 - 4$ from $x = -2$ to $x = 2$
    (b) $y = x^3 - x$ from $x = -1$ to $x = 1$
    (c) $y = \sin x$ from $x = 0$ to $x = 2\pi$
    (d) $y = e^x - 1$ from $x = 0$ to $x = \ln 2$
23. Find the area between these curves:
    (a) $y = x^2$ and $y = 4$ from $x = 0$ to $x = 2$
    (b) $y = x^2$ and $y = 2x + 3$ from $x = -1$ to $x = 3$
    (c) $y = \sin x$ and $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{4}$
    (d) $y = e^x$ and $y = 1$ from $x = 0$ to $x = 1$
24. Find the total area enclosed by:
    (a) $y = x^2 - 1$ and the x-axis
    (b) $y = x^3 - 4x$ and the x-axis
    (c) $y = \sin x$ and $y = 0$ from $x = 0$ to $x = 2\pi$
    (d) $y = x^2 - 2x - 3$ and the x-axis
25. A region is bounded by $y = x^2$, $y = 0$, $x = 1$, and $x = 3$.
    (a) Calculate the area of the region
    (b) Find the x-coordinate of the centroid
    (c) Calculate the moment about the y-axis
    (d) Find the average value of $y = x^2$ over $[1, 3]$

## Section F: Fundamental Theorem of Calculus

26. Use the fundamental theorem to evaluate:
    (a) $\frac{d}{dx} \int_0^x t^2 \, dt$
    (b) $\frac{d}{dx} \int_1^x \sin t \, dt$
    (c) $\frac{d}{dx} \int_0^{x^2} e^t \, dt$
    (d) $\frac{d}{dx} \int_{x^2}^{x^3} \cos t \, dt$
27. Find these derivatives:
    (a) $\frac{d}{dx} \int_0^x \sqrt{1 + t^2} \, dt$
    (b) $\frac{d}{dx} \int_x^2 \frac{1}{t} \, dt$
    (c) $\frac{d}{dx} \int_{\sin x}^{\cos x} t^2 \, dt$
    (d) $\frac{d}{dx} \int_0^{x^2} \sin(t^2) \, dt$
28. Given $F(x) = \int_0^x f(t) \, dt$ where $f$ is continuous:
    (a) Prove that $F'(x) = f(x)$
    (b) If $f(x) = x^2 + 1$, find $F(x)$
    (c) Verify that $F'(x) = f(x)$ for your answer
    (d) Calculate $F(2) - F(1)$ and interpret geometrically

29. Solve these differential equations using antiderivatives:
    (a) $\frac{dy}{dx} = 3x^2 + 2x - 1$ with $y(0) = 5$
    (b) $\frac{dy}{dx} = e^x + \sin x$ with $y(0) = 1$
    (c) $\frac{d^2y}{dx^2} = 6x + 2$ with $y'(0) = 1$ and $y(0) = 0$
    (d) $\frac{dy}{dx} = \frac{1}{x}$ with $y(1) = 2$
30. For the function $g(x) = \int_1^x \frac{1}{t}\,dt$:
    (a) Find $g'(x)$
    (b) Show that $g(xy) = g(x) + g(y)$ for $x, y > 0$
    (c) Prove that $g(x^n) = n \cdot g(x)$ for $x > 0$ and integer $n$
    (d) Identify $g(x)$ in terms of elementary functions

## Section G: Volumes of Revolution

31. Find the volume when these curves are rotated about the x-axis:
    (a) $y = x$ from $x = 0$ to $x = 2$
    (b) $y = x^2$ from $x = 0$ to $x = 1$
    (c) $y = \sqrt{x}$ from $x = 0$ to $x = 4$
    (d) $y = e^x$ from $x = 0$ to $x = 1$
32. Calculate volumes of revolution about the x-axis:
    (a) $y = 2x + 1$ from $x = 0$ to $x = 3$
    (b) $y = x^2 + 1$ from $x = -1$ to $x = 1$
    (c) $y = \sin x$ from $x = 0$ to $x = \pi$
    (d) $y = \frac{1}{x}$ from $x = 1$ to $x = 2$
33. Find volumes when rotated about the y-axis:
    (a) $x = y^2$ from $y = 0$ to $y = 2$
    (b) $x = \sqrt{y}$ from $y = 0$ to $y = 4$
    (c) $x = e^y$ from $y = 0$ to $y = 1$
    (d) $x = \ln y$ from $y = 1$ to $y = e$
34. Use the washer method to find volumes:
    (a) Region between $y = x^2$ and $y = 4$ rotated about x-axis
    (b) Region between $y = x$ and $y = x^2$ rotated about x-axis
    (c) Region between $y = e^x$ and $y = 1$ from $x = 0$ to $x = 1$ rotated about x-axis
    (d) Region between $y = \sqrt{x}$ and $y = x$ rotated about y-axis
35. A solid has circular cross-sections. The radius at height $h$ is $r(h) = \sqrt{4 - h^2}$ for $0 \leq h \leq 2$.
    (a) Set up the integral for the volume
    (b) Calculate the volume
    (c) Identify the shape of the solid
    (d) Find the surface area if this represents a hemisphere

## Section H: Applications in Physics and Engineering

36. A particle moves with velocity $v(t) = 3t^2 - 6t + 2$ m/s.
    (a) Find the displacement from $t = 0$ to $t = 3$
    (b) Calculate the total distance traveled
    (c) Find the position function if $s(0) = 5$
    (d) Determine when the particle changes direction
    (e) Calculate the average velocity over $[0, 3]$
37. The acceleration of an object is $a(t) = 6t - 4$ m/s².
    (a) Find the velocity if $v(0) = 2$ m/s
    (b) Find the position if $s(0) = 0$
    (c) Calculate the displacement from $t = 1$ to $t = 3$
    (d) Find when the object is at rest
    (e) Determine the maximum distance from the origin
38. Water flows into a tank at rate $R(t) = 5 + 2t$ liters per minute.
    (a) Find the total volume added in the first 10 minutes
    (b) If the tank initially contains 50 liters, find $V(t)$
    (c) Calculate the average flow rate over 10 minutes
    (d) Find when the tank contains 200 liters
    (e) Determine the rate of change of flow rate
39. The force on a spring is $F(x) = kx$ where $x$ is displacement from equilibrium.
    (a) Find the work done stretching the spring from $x = 0$ to $x = a$
    (b) If $k = 100$ N/m, calculate work to stretch 0.5 m
    (c) Find the potential energy stored in the spring
    (d) Compare with gravitational potential energy $mgh$
40. The rate of heat conduction is $\frac{dQ}{dt} = -kA\frac{dT}{dx}$ (Fourier's law).
    (a) If temperature varies as $T(x) = 100 - 2x^2$, find $\frac{dT}{dx}$
    (b) Calculate the heat flux at $x = 5$
    (c) Find the total heat conducted through a rod from $x = 0$ to $x = 10$
    (d) Interpret the negative sign physically

## Section I: Advanced Applications and Techniques

41. The center of mass of a thin rod from $x = a$ to $x = b$ with density $\rho(x)$ is: $\bar{x} = \frac{\int_a^b x\rho(x)\,dx}{\int_a^b \rho(x)\,dx}$
    (a) Find the center of mass of a rod from $x = 0$ to $x = 2$ with density $\rho(x) = x + 1$
    (b) Calculate the total mass of the rod
    (c) Find the center of mass if density is $\rho(x) = e^x$
    (d) Compare with uniform density $\rho(x) = 1$
42. The moment of inertia about the x-axis is $I_x = \int y^2\,dm$ where $dm = \rho\,dA$.

(a) Find $I_x$ for the region under $y = x^2$ from $x = 0$ to $x = 1$ with uniform density
(b) Calculate the radius of gyration $r_g = \sqrt{\frac{I_x}{M}}$
(c) Find the moment of inertia about the y-axis
(d) Compare $I_x$ and $I_y$ and explain physically

43. Arc length of a curve $y = f(x)$ from $x = a$ to $x = b$ is: $L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$
    (a) Find the arc length of $y = x^2$ from $x = 0$ to $x = 1$
    (b) Calculate the arc length of $y = \ln x$ from $x = 1$ to $x = e$
    (c) Find the perimeter of one arch of $y = \sin x$
    (d) Derive the formula for arc length using differential geometry

44. Surface area of revolution about x-axis is: $S = 2\pi \int_a^b y\sqrt{1 + (y')^2}\, dx$
    (a) Find the surface area when $y = x$ from $x = 0$ to $x = 1$ is rotated
    (b) Calculate surface area for $y = \sqrt{x}$ from $x = 0$ to $x = 4$
    (c) Find the surface area of a sphere of radius $r$
    (d) Compare with the volume formula for verification

45. Economic applications of integration:
    (a) If marginal cost is $MC(x) = 2x + 3$, find total cost function given fixed costs of £100
    (b) Calculate consumer surplus if demand is $p = 20 - x^2$ and price is £4
    (c) Find producer surplus for supply curve $p = x^2 + 1$ at equilibrium price £5
    (d) Determine the total economic surplus at market equilibrium

46. Probability density functions satisfy $\int_{-\infty}^{\infty} f(x)\, dx = 1$.
    (a) Find the constant $k$ so that $f(x) = kx^2$ is a PDF on $[0, 2]$
    (b) Calculate $P(X \leq 1)$ for this distribution
    (c) Find the mean $\mu = \int x f(x)\, dx$
    (d) Calculate the variance $\sigma^2 = \int (x - \mu)^2 f(x)\, dx$

47. Design an integration problem modeling a real-world scenario:
    (a) Define your physical or geometric setup clearly
    (b) Set up the appropriate integral(s)
    (c) Evaluate the integral(s) analytically
    (d) Interpret your results in the original context
    (e) Discuss limitations and assumptions of your model

48. Numerical integration and error analysis:
    (a) Use the trapezoidal rule with $n = 4$ to approximate $\int_0^1 e^{x^2}\, dx$
    (b) Apply Simpson's rule with $n = 4$ to the same integral
    (c) Compare your approximations and estimate the error
    (d) Explain why this integral cannot be evaluated analytically
    (e) Research applications where numerical integration is essential

**Answer Space**

Use this space for your working and answers.

**END OF TEST**

Total marks: 150

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