A Level Pure Mathematics
Practice Test 2: Differentiation

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Fundamental Polynomial Differentiation

1. Find the derivatives of these polynomial functions:
   (a) $f(x) = 4x^5 - 3x^4 + 2x^3 - x^2 + 6$
   (b) $g(x) = 3x^4 + \frac{2}{3}x^2 - 5x + 2$
   (c) $h(x) = (x + 2)(x - 4)$
   (d) $k(x) = (3x + 1)^2$
   (e) $p(x) = x^2(x^3 + 2)$
   (f) $q(x) = \frac{x^3 - 3x + 2}{x}$
2. Calculate $\frac{dy}{dx}$ for:
   (a) $y = 4x^{-3} + 3x^{-1} - 5$
   (b) $y = \frac{3}{x^2} - \frac{4}{\sqrt{x}} + 2\sqrt{x}$
   (c) $y = 3\sqrt{x^5} + \frac{2}{x^3} - x^{-\frac{3}{4}}$
   (d) $y = (2x - \frac{1}{x})^2$
3. Determine the gradient at the specified points:
   (a) $y = x^4 - 2x^3 + 3x - 1$ at $x = 1$
   (b) $y = 3x^3 - 4x^2 + 2$ at $x = -1$
   (c) $y = \frac{x^3 - 1}{x}$ at $x = 2$
   (d) $y = (x + 1)^3$ at $x = 0$
4. Find equations of tangent lines to:
   (a) $y = x^4 - 3x^2 + 2x$ at the point where $x = 2$
   (b) $y = 3x^2 - 4x + 1$ at the point $(1, 0)$
   (c) $y = x^3 - 2x$ at the point where the gradient is 1
   (d) $y = \frac{x^3}{3} - 2x + 4$ at the point where $x = 3$
5. For $f(x) = px^3 + qx^2 + rx + s$ where $f'(x) = 9x^2 - 6x + 2$:
   (a) Determine the values of p, q, and r
   (b) If $f(1) = 8$, find the value of s
   (c) Write the complete function $f(x)$
   (d) Calculate $f(3)$ and $f'(0)$

Section B: Exponential, Logarithmic and Trigonometric Functions

6. Differentiate these exponential and logarithmic functions:
   (a) $f(x) = 2e^x$
   (b) $g(x) = e^x - 3x^3$
   (c) $h(x) = x^2 e^x$
   (d) $k(x) = 2\ln x$
   (e) $p(x) = x^2 \ln x$
   (f) $q(x) = \frac{e^x}{x}$
7. Find derivatives of these trigonometric functions:
   (a) $f(x) = 2\sin x - \cos x$
   (b) $g(x) = 4\cos x + \sin x - x^3$
   (c) $h(x) = x^2 \cos x$
   (d) $k(x) = \frac{\sin x}{x}$
   (e) $p(x) = \cot x$
   (f) $q(x) = \csc x$
8. Calculate $\frac{dy}{dx}$ for:
   (a) $y = e^{3x}$
   (b) $y = \ln(2x)$
   (c) $y = \sin(3x)$
   (d) $y = \cos(3x + 2)$
   (e) $y = e^{x^3}$
   (f) $y = \ln(x^3 - 2)$
9. Use combination rules to differentiate:
   (a) $f(x) = e^x \sin x$
   (b) $g(x) = x^3 \cos x$
   (c) $h(x) = \frac{x}{e^x}$
   (d) $k(x) = \frac{\cos x}{\sin x}$
   (e) $p(x) = (\ln x)^3$
   (f) $q(x) = \sqrt{\cos x}$
10. Find the derivatives of:
   (a) $f(x) = \cos^2 x$
   (b) $g(x) = \sin^4 x$
   (c) $h(x) = e^{\cos x}$
   (d) $k(x) = \ln(\sin x)$
   (e) $p(x) = (\sin x - \cos x)^2$
   (f) $q(x) = \sin^{-1} x$ (inverse sine)

Section C: Product and Quotient Rules

11. Apply the product rule to differentiate:
    (a) $f(x) = (x^3 - 2)(x^2 + 3)$
    (b) $g(x) = (3x + 1)(x^3 - x + 2)$
    (c) $h(x) = x^3 e^x$
    (d) $k(x) = (2x - 1)\ln x$
    (e) $p(x) = \cos x \sin x$
    (f) $q(x) = x^2 \cos x$

12. Use the quotient rule to find derivatives:
    (a) $f(x) = \frac{x^3-1}{x+2}$
    (b) $g(x) = \frac{3x-2}{x^2-1}$
    (c) $h(x) = \frac{x}{e^x}$
    (d) $k(x) = \frac{e^x}{x-1}$
    (e) $p(x) = \frac{\cos x}{1+\sin x}$
    (f) $q(x) = \frac{x^3}{\cos x}$

13. Select the best method and differentiate:
    (a) $f(x) = \frac{x^2-3x}{x}$
    (b) $g(x) = (x^3 + 1)(x - 3)$
    (c) $h(x) = \frac{x^3-2x+1}{x^3}$
    (d) $k(x) = x^2(x^2 - 1)^3$
    (e) $p(x) = \frac{(2x-1)^2}{x}$
    (f) $q(x) = x^3\sqrt{x - 2}$

14. Given $f(x) = x^3$ and $g(x) = \cos x$:
    (a) Calculate $(fg)'(x)$ using the product rule
    (b) Find $(\frac{f}{g})'(x)$ using the quotient rule
    (c) Evaluate $(fg)'(\frac{\pi}{3})$
    (d) Calculate $(\frac{f}{g})'(\frac{\pi}{4})$

15. Verify these differentiation formulas:
    (a) Constant multiple rule: $(cf)' = cf'$
    (b) Sum rule: $(f + g)' = f' + g'$
    (c) Show that $(\frac{k}{v})' = -\frac{kv'}{v^2}$ where k is constant
    (d) Prove that $(f^2)' = 2ff'$

Section D: Chain Rule Applications

16. Use the chain rule to differentiate:
    (a) $f(x) = (3x - 2)^4$
    (b) $g(x) = (x^3 + 2x - 1)^5$
    (c) $h(x) = \sqrt{x^3 - 1}$

(d) $k(x) = (2x+3)^{-3}$
(e) $p(x) = \cos(3x-1)$
(f) $q(x) = \sin(x^3)$

17. Calculate $\frac{dy}{dx}$ for:
(a) $y = e^{2x+3}$
(b) $y = \ln(3x-1)$
(c) $y = (x^3 - 2x)^4$
(d) $y = \cos^2 x$
(e) $y = \sin(e^x)$
(f) $y = e^{\cos x}$

18. Differentiate these composite functions:
(a) $f(x) = (e^x - 1)^2$
(b) $g(x) = \ln(x^3 + x + 1)$
(c) $h(x) = \cos(\ln x)$
(d) $k(x) = e^{x \sin x}$
(e) $p(x) = (\cos x - \sin x)^3$
(f) $q(x) = \ln(\cos x)$

19. Combine multiple rules to differentiate:
(a) $f(x) = x^2(3x-1)^4$
(b) $g(x) = \frac{x}{(2x+1)^3}$
(c) $h(x) = x\sin(2x)$
(d) $k(x) = e^x \sin(3x)$
(e) $p(x) = \frac{e^x}{\sqrt{x}}$
(f) $q(x) = \frac{(x^3-1)^2}{x}$

20. Find second derivatives of:
(a) $f(x) = (2x-1)^3$
(b) $g(x) = \cos(3x)$
(c) $h(x) = e^{-2x}$
(d) $k(x) = \ln(x^3)$
(e) $p(x) = x \sin x$
(f) $q(x) = e^x \cos x$

Section E: Critical Points and Optimization

21. Locate all stationary points for:
(a) $f(x) = x^3 - 6x^2 + 9x + 1$
(b) $g(x) = 3x^3 - 9x^2 + 6x + 2$
(c) $h(x) = x^4 - 8x^2 + 10$
(d) $k(x) = \frac{x^3}{x+1}$ for $x \neq -1$

22. Classify stationary points using the second derivative test:
    (a) $f(x) = x^3 - 9x^2 + 24x - 5$
    (b) $g(x) = 3x^3 - 4x^2 - 6x + 8$
    (c) $h(x) = x^4 - 4x^2 + 5$
    (d) $k(x) = x^2e^{-x}$

23. Find and analyze all critical points:
    (a) $f(x) = x^3 + 3x^2 - 9x + 2$
    (b) $g(x) = 2x^3 - 15x^2 + 24x - 3$
    (c) $h(x) = x^4 - 4x^3 + 4x^2 + 1$
    (d) $k(x) = x^2 - \frac{8}{x}$ for $x > 0$

24. For the function $f(x) = px^3 + qx^2 + rx + s$:
    (a) What conditions ensure exactly one stationary point exists?
    (b) If $f(x) = x^3 + 3x^2 + 3x + 1$, verify it has one stationary point
    (c) Find values of m for which $f(x) = x^3 - 3mx^2 + 1$ has a local maximum at $x = 2$

25. Examine the function $f(x) = \frac{x^2+1}{x}$:
    (a) Determine the domain of $f(x)$
    (b) Calculate $f'(x)$ and find stationary points
    (c) Classify the nature of stationary points
    (d) Identify all asymptotes
    (e) Sketch the complete graph

Section F: Related Rates and Motion

26. A particle moves with position $s(t) = 2t^3 - 9t^2 + 12t - 3$ meters at time t seconds.
    (a) Determine velocity $v(t)$ and acceleration $a(t)$
    (b) Find when the particle is stationary
    (c) Calculate velocity and acceleration at $t = 3$
    (d) When is the acceleration zero?
    (e) Find total distance traveled between $t = 0$ and $t = 3$

27. For a cube with side length s, the volume is $V = s^3$. If the side increases at 3 cm/s:
    (a) Find the rate of volume change when $s = 4$ cm
    (b) Express $\frac{dV}{dt}$ in terms of s and $\frac{ds}{dt}$
    (c) When is volume increasing at 150 cm^3/s?
    (d) Find the rate of surface area change when $s = 6$ cm

28. A 10-meter ladder leans against a wall. The base moves away at 2 m/s.
    (a) Establish the relationship between distances
    (b) Find how fast the top descends when the base is 6m from the wall
    (c) Calculate the rate of angle change with the horizontal
    (d) When does the top descend fastest?

29. Sand pours into a cone-shaped pile at 3 m^3/min. The height equals the radius.
   (a) Express volume in terms of height h
   (b) Find how fast height increases when $h = 2$m
   (c) Calculate the rate of radius change when $h = 3$m
   (d) When does the height increase fastest?

30. A city's population follows $P(t) = 50000e^{0.03t}$ where t is years.
   (a) Find the growth rate $\frac{dP}{dt}$
   (b) Calculate population and growth rate after 3 years
   (c) When is the population growing at 2000 people per year?
   (d) Express growth rate as percentage of current population

Section G: Optimization Applications

31. A rancher has 300m of fencing for a rectangular corral adjacent to a barn (no fence needed against barn).
   (a) Express area as a function of one dimension
   (b) Find dimensions for maximum area
   (c) Calculate the maximum area
   (d) Verify this is indeed a maximum

32. A cylindrical can with lid has volume 1000 cm^3. Material costs: base £4/m^2, sides £2/m^2, top £3/m^2.
   (a) Express total cost in terms of radius
   (b) Find dimensions for minimum cost
   (c) Calculate the minimum cost
   (d) Determine the height-to-radius ratio

33. A company's revenue function is $R(x) = -x^3 + 15x^2 + 48x - 80$ thousand pounds for x thousand units produced.
   (a) Find production levels for maximum and minimum revenue
   (b) Calculate the maximum revenue
   (c) Determine the marginal revenue function
   (d) Find optimal production level

34. An arched doorway consists of a rectangle with a semicircular top. The perimeter is 16m.
   (a) Express area in terms of rectangle width
   (b) Find dimensions for maximum area
   (c) Calculate maximum area
   (d) Find ratio of rectangle height to width

35. A right circular cone is inscribed in a sphere of radius 8 cm. Find dimensions to maximize volume.
   (a) Express cone volume in terms of cone height
   (b) Find critical points
   (c) Determine optimal height and radius
   (d) Calculate maximum volume
   (e) Verify this gives maximum volume

Section H: Implicit Differentiation and Advanced Techniques

36. Use implicit differentiation to find $\frac{dy}{dx}$:
    (a) $x^2 + y^2 = 16$
    (b) $x^2 - 2xy + y^2 = 9$
    (c) $x^3 - y^3 = 3xy$
    (d) $\cos(xy) = x - y$
    (e) $e^{xy} = x^2 + y$
    (f) $\ln(x + y) = x - y$
37. Find tangent line equations at given points:
    (a) $x^2 + y^2 = 25$ at $(3, 4)$
    (b) $x^2 + xy + y^2 = 3$ at $(1, 1)$
    (c) $x^3 - y^3 = 8$ at $(2, 0)$
    (d) $ye^x = 3$ at $(0, 3)$
38. Find $\frac{d^2y}{dx^2}$ using implicit differentiation:
    (a) $x^2 + y^2 = 4$
    (b) $xy = 4$
    (c) $x^2 - y^2 = 1$
39. Two ships leave port simultaneously. Ship A sails north at 15 km/h, Ship B sails east at 20 km/h.
    (a) Express distance between ships as function of time
    (b) Find separation rate after 3 hours
    (c) When are they separating at 25 km/h?
    (d) Calculate minimum separation distance
40. A spherical balloon expands so radius increases at 0.5 cm/s. Find rate of change of:
    (a) Volume when $r = 8$ cm
    (b) Surface area when $r = 12$ cm
    (c) Radius when volume is 2000 cm³
    (d) Volume when surface area is 300π cm²

Section I: Complex Applications and Modeling

41. A Gothic window has rectangular base topped by an equilateral triangle, with total perimeter 24m.
    (a) Find dimensions to maximize area
    (b) Calculate maximum area
    (c) Determine optimal ratio of triangle side to rectangle width
    (d) Find what fraction of area is triangular
42. The stiffness of a rectangular beam varies as wd^3 where w is width and d is depth. Cut from circular log of diameter 20 cm.

(a) Express stiffness in terms of width w
(b) Find dimensions for maximum stiffness
(c) Calculate ratio $\frac{d}{w}$ for stiffest beam
(d) Compare with circular cross-section beam

43. Blood alcohol concentration follows $C(t) = \frac{0.2t}{t^2+1}$ percent where t is hours after drinking.
    (a) Find when concentration peaks
    (b) Calculate maximum concentration
    (c) Determine rate of change at $t = 2$
    (d) When is concentration decreasing fastest?
    (e) Find when concentration drops to half its peak

44. An isosceles triangle is inscribed in a circle of radius 10 cm with base as chord.
    (a) Express area in terms of base length
    (b) Find base length for maximum area
    (c) Calculate maximum area
    (d) Show optimal triangle is equilateral

45. A manufacturer's profit per unit is $P(x) = 60 - 0.02x$ pounds and cost per unit is $C(x) = 20 + 0.01x$ pounds for x units.
    (a) Find total profit function
    (b) Determine production level for maximum total profit
    (c) Calculate maximum total profit
    (d) Find break-even production levels
    (e) Analyze economic interpretation

46. For two positive numbers with product 100:
    (a) Minimize their sum
    (b) Maximize the sum of their reciprocals
    (c) Minimize $x + 4y$ where $xy = 100$
    (d) Maximize $x^2 + y^2$ subject to $xy = 100$
    (e) Explain the different optimization strategies

47. A satellite's orbital altitude follows $h(t) = 400 + 50\sin(\frac{2\pi t}{90})$ km where t is minutes.
    (a) Find rate of altitude change
    (b) Calculate maximum rate of ascent
    (c) Determine when satellite climbs fastest
    (d) Find average altitude over one orbit
    (e) Analyze orbital characteristics

48. Design an optimization problem for environmental sustainability:
    (a) Define your scenario and constraints clearly
    (b) Formulate objective function and variables
    (c) Apply calculus methods to find optimal solution
    (d) Verify solution satisfies all constraints
    (e) Discuss real-world implementation challenges

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

For more resources and practice materials, visit:
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