A Level Pure Mathematics
Practice Test 5: Integration

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Basic Integration - Polynomials

1. Find these indefinite integrals:
   (a) $\int(7x^2 - 6x + 8)\,dx$
   (b) $\int(5x^3 - 2x^2 + 4x - 3)\,dx$
   (c) $\int(6x^4 + x - 8)\,dx$
   (d) $\int(3x^2 + \frac{4}{5}x - 6)\,dx$
   (e) $\int(5x - 4)^2\,dx$
   (f) $\int(4x + 3)(x - 2)\,dx$

2. Integrate these functions involving negative and fractional powers:
   (a) $\int x^{-6}\,dx$
   (b) $\int(6x^{-1} + 3x^{\frac{4}{5}})\,dx$
   (c) $\int \frac{5}{x^7}\,dx$
   (d) $\int \sqrt[6]{x}\,dx$
   (e) $\int \frac{6}{\sqrt{x}}\,dx$
   (f) $\int(4x^{\frac{5}{3}} - 5x^{-\frac{4}{5}})\,dx$

3. Find these integrals by expanding first:
   (a) $\int \frac{5x^3+3x^2-4x}{x}\,dx$
   (b) $\int \frac{x^2-36}{x}\,dx$
   (c) $\int \frac{(3x+2)^2}{x}\,dx$
   (d) $\int \frac{4x^3-27}{x^2}\,dx$

4. Evaluate these definite integrals:
   (a) $\int_1^3(3x^2 - 4x + 2)\,dx$
   (b) $\int_0^5(6x + 3)\,dx$
   (c) $\int_{-3}^2 x^3\,dx$
   (d) $\int_1^{25} \sqrt{x}\,dx$

5. Find the function $f(x)$ given:
   (a) $f'(x) = 10x^2 - 8x + 5$ and $f(0) = 8$
   (b) $f'(x) = 14x + 3$ and $f(1) = 12$
   (c) $f''(x) = 6x + 8$, $f'(0) = 5$, and $f(0) = 7$
   (d) $f'(x) = \frac{5}{x^6}$ for $x > 0$ and $f(1) = 4$

Section B: Integration of Standard Functions

6. Integrate these exponential and logarithmic functions:
   (a) $\int 9e^x \, dx$
   (b) $\int 10e^x \, dx$
   (c) $\int e^{6x} \, dx$
   (d) $\int e^{-5x} \, dx$
   (e) $\int \frac{6}{x} \, dx$ for $x > 0$
   (f) $\int \frac{8}{x} \, dx$
7. Integrate these trigonometric functions:
   (a) $\int 9 \sin x \, dx$
   (b) $\int 8 \cos x \, dx$
   (c) $\int 10 \sin x \, dx$
   (d) $\int 6 \cos x \, dx$
   (e) $\int 6 \sec^2 x \, dx$
   (f) $\int 5 \text{cosec}^2 x \, dx$
8. Find these integrals:
   (a) $\int (5 \sin x + 4 \cos x) \, dx$
   (b) $\int (6e^x - 3x^3) \, dx$
   (c) $\int (5e^x - 6 \sin x) \, dx$
   (d) $\int \left(\frac{5}{x} + 4x\right) \, dx$ for $x > 0$
   (e) $\int (6 \cos x + 4e^{-x}) \, dx$
   (f) $\int \left(5x^2 + \frac{6}{x^2}\right) \, dx$ for $x > 0$
9. Evaluate these definite integrals:
   (a) $\int_0^{5\pi} \sin x \, dx$
   (b) $\int_0^{\frac{\pi}{3}} \cos x \, dx$
   (c) $\int_0^5 e^x \, dx$
   (d) $\int_1^{e^5} \frac{1}{x} \, dx$
   (e) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sec^2 x \, dx$
   (f) $\int_0^{\ln 6} e^{-x} \, dx$
10. Find the exact values:
    (a) $\int_0^{\frac{\pi}{4}} 6 \sin x \, dx$
    (b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos x \, dx$
    (c) $\int_0^{\ln 6} 4e^x \, dx$
    (d) $\int_1^{e^4} \frac{6}{x} \, dx$

Section C: Integration by Substitution

11. Use substitution to find these integrals:
    (a) $\int (6x+5)^5 \, dx$
    (b) $\int (4x-7)^4 \, dx$
    (c) $\int x(5x^2+3)^4 \, dx$
    (d) $\int x\sqrt{4x^2+5} \, dx$
    (e) $\int \frac{5x}{4x^2-3} \, dx$
    (f) $\int \frac{6x}{(4x^2+5)^2} \, dx$
12. Find these integrals using appropriate substitutions:
    (a) $\int \sin(6x+2) \, dx$
    (b) $\int \cos(4x-\frac{\pi}{3}) \, dx$
    (c) $\int e^{6x+4} \, dx$
    (d) $\int e^{-6x} \, dx$
    (e) $\int \frac{1}{6x+5} \, dx$
    (f) $\int \frac{5}{4x-9} \, dx$
13. Use substitution for these more complex integrals:
    (a) $\int x^2(3x^3+7)^5 \, dx$
    (b) $\int \frac{x^2}{\sqrt{4x^3+3}} \, dx$
    (c) $\int xe^{5x^2} \, dx$
    (d) $\int \frac{\ln x}{5x} \, dx$
    (e) $\int \sin 5x \cos 4x \, dx$
    (f) $\int \sec 2x \tan 2x \, dx$
14. Evaluate these definite integrals using substitution:
    (a) $\int_0^3 x(2x^2+3)^3 \, dx$
    (b) $\int_0^{\frac{\pi}{8}} \sin 5x \cos 3x \, dx$
    (c) $\int_2^4 \frac{4x}{3x^2+2} \, dx$
    (d) $\int_0^3 xe^{4x^2} \, dx$
15. Find these integrals by recognizing the derivative pattern:
    (a) $\int \frac{10x+4}{5x^2+4x-2} \, dx$
    (b) $\int \frac{12x^2+8}{4x^3+8x-3} \, dx$
    (c) $\int \frac{5e^x}{e^x-4} \, dx$
    (d) $\int \frac{4\sin x}{\cos x} \, dx$

Section D: Integration by Parts

16. Use integration by parts to find:
   (a) $\int 5xe^x \, dx$
   (b) $\int 4x \sin x \, dx$
   (c) $\int 4x \cos x \, dx$
   (d) $\int x^2 e^{5x} \, dx$
   (e) $\int 5x \ln x \, dx$
   (f) $\int e^x \sin 4x \, dx$
17. Apply integration by parts to:
   (a) $\int 5 \ln x \, dx$
   (b) $\int x^5 \ln x \, dx$
   (c) $\int 4x \ln x \, dx$
   (d) $\int \ln(5x + 2) \, dx$
   (e) $\int 3x \tan^{-1} x \, dx$
   (f) $\int x^2 \sin 4x \, dx$
18. Find these integrals that may require multiple applications:
   (a) $\int x^2 e^{-5x} \, dx$
   (b) $\int x^2 \cos 4x \, dx$
   (c) $\int e^{5x} \cos 4x \, dx$
   (d) $\int e^{5x} \sin 4x \, dx$
   (e) $\int \sin(\ln 4x) \, dx$
   (f) $\int x^3 e^{5x} \, dx$
19. Evaluate these definite integrals:
   (a) $\int_0^5 xe^x \, dx$
   (b) $\int_0^{\frac{\pi}{4}} x \sin x \, dx$
   (c) $\int_1^{e^5} x \ln x \, dx$
   (d) $\int_0^{\frac{\pi}{8}} x \cos 4x \, dx$
20. Prove these reduction formulas using integration by parts:
   (a) $I_n = \int x^n e^{5x} \, dx = \frac{x^n e^{5x}}{5} - \frac{n}{5} I_{n-1}$
   (b) $I_n = \int \cot^n x \, dx = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$ for $n \geq 2$
   (c) Use the first formula to find $\int x^5 e^{5x} \, dx$

Section E: Area Under Curves

21. Find the area under these curves:
   (a) $y = 5x^2$ from $x = 0$ to $x = 3$
   (b) $y = 6x + 4$ from $x = 0$ to $x = 3$
   (c) $y = x^3 + 4x$ from $x = 0$ to $x = 2$
   (d) $y = 4 \sin x$ from $x = 0$ to $x = \frac{\pi}{4}$

22. Calculate the area between the curve and the x-axis:
    (a) $y = x^2 - 25$ from $x = -5$ to $x = 5$
    (b) $y = x^3 - 16x$ from $x = -4$ to $x = 4$
    (c) $y = 4\cos x$ from $x = 0$ to $x = 2\pi$
    (d) $y = e^x - 5$ from $x = 0$ to $x = \ln 6$
23. Find the area between these curves:
    (a) $y = 5x^2$ and $y = 20$ from $x = 0$ to $x = 2$
    (b) $y = x^2$ and $y = 5x - 4$ from $x = 1$ to $x = 4$
    (c) $y = \sin 4x$ and $y = \cos 3x$ from $x = 0$ to $x = \frac{\pi}{8}$
    (d) $y = 4e^x$ and $y = 8$ from $x = 0$ to $x = \ln 2$
24. Find the total area enclosed by:
    (a) $y = x^2 - 25$ and the x-axis
    (b) $y = x^3 - 36x$ and the x-axis
    (c) $y = 4\sin x$ and $y = 0$ from $x = 0$ to $x = 2\pi$
    (d) $y = x^2 - 5x - 6$ and the x-axis
25. A region is bounded by $y = 5x^2$, $y = 0$, $x = 2$, and $x = 4$.
    (a) Calculate the area of the region
    (b) Find the x-coordinate of the centroid
    (c) Calculate the moment about the y-axis
    (d) Find the average value of $y = 5x^2$ over $[2, 4]$

Section F: Fundamental Theorem of Calculus

26. Use the fundamental theorem to evaluate:
    (a) $\frac{d}{dx} \int_0^x 5t^2 \, dt$
    (b) $\frac{d}{dx} \int_5^x \sin t \, dt$
    (c) $\frac{d}{dx} \int_0^{5x} e^t \, dt$
    (d) $\frac{d}{dx} \int_{4x}^{x^2} \cos t \, dt$
27. Find these derivatives:
    (a) $\frac{d}{dx} \int_0^x \sqrt{25 + t^2} \, dt$
    (b) $\frac{d}{dx} \int_x^6 \frac{5}{t} \, dt$
    (c) $\frac{d}{dx} \int_{\sin 4x}^{\cos 3x} t^4 \, dt$
    (d) $\frac{d}{dx} \int_0^{x^4} \sin(t^5) \, dt$
28. Given $L(x) = \int_4^x f(t) \, dt$ where f is continuous:
    (a) Prove that $L'(x) = f(x)$
    (b) If $f(x) = 5x^2 + 4$, find $L(x)$
    (c) Verify that $L'(x) = f(x)$ for your answer
    (d) Calculate $L(6) - L(5)$ and interpret geometrically

29. Solve these differential equations using antiderivatives:
    (a) $\frac{dy}{dx} = 10x^3 - 8x + 4$ with $y(0) = 6$
    (b) $\frac{dy}{dx} = 5e^x + \sin x$ with $y(0) = 4$
    (c) $\frac{d^2y}{dx^2} = 6x - 12$ with $y'(0) = 5$ and $y(0) = 4$
    (d) $\frac{dy}{dx} = \frac{5}{x}$ with $y(1) = 6$

30. For the function $l(x) = \int_5^x \frac{1}{t}\,dt$:
    (a) Find $l'(x)$
    (b) Show that $l(xy) = l(x) + l(y)$ for $x, y > 0$
    (c) Prove that $l(x^n) = n \cdot l(x)$ for $x > 0$ and integer n
    (d) Express $l(x)$ in terms of elementary functions

Section G: Volumes of Revolution

31. Find the volume when these curves are rotated about the x-axis:
    (a) $y = 5x$ from $x = 0$ to $x = 3$
    (b) $y = 4x^2$ from $x = 0$ to $x = 3$
    (c) $y = \sqrt{5x}$ from $x = 0$ to $x = 5$
    (d) $y = e^{5x}$ from $x = 0$ to $x = 1$

32. Calculate volumes of revolution about the x-axis:
    (a) $y = 4x + 2$ from $x = 0$ to $x = 3$
    (b) $y = x^2 + 4$ from $x = -2$ to $x = 2$
    (c) $y = 4\sin x$ from $x = 0$ to $x = \frac{\pi}{3}$
    (d) $y = \frac{5}{x}$ from $x = 1$ to $x = 5$

33. Find volumes when rotated about the y-axis:
    (a) $x = 5y^2$ from $y = 0$ to $y = 2$
    (b) $x = \sqrt{5y}$ from $y = 0$ to $y = 5$
    (c) $x = e^{5y}$ from $y = 0$ to $y = 1$
    (d) $x = 5\ln y$ from $y = 1$ to $y = e^5$

34. Use the washer method to find volumes:
    (a) Region between $y = 4x^2$ and $y = 16$ rotated about x-axis
    (b) Region between $y = 5x$ and $y = x^2$ rotated about x-axis
    (c) Region between $y = 4e^x$ and $y = 5$ from $x = 0$ to $x = \ln(\frac{5}{4})$ rotated about x-axis
    (d) Region between $y = \sqrt{5x}$ and $y = 4x$ rotated about y-axis

35. A solid has circular cross-sections. The radius at height h is $r(h) = \sqrt{36 - h^2}$ for $0 \leq h \leq 6$.
    (a) Set up the integral for the volume
    (b) Calculate the volume
    (c) Identify the shape of the solid
    (d) Find the surface area if this represents a hemisphere

Section H: Applications in Physics and Engineering

36. A particle moves with velocity $v(t) = 4t^2 - 10t + 6$ m/s.
    (a) Find the displacement from $t = 0$ to $t = 5$
    (b) Calculate the total distance traveled
    (c) Find the position function if $s(0) = 15$
    (d) Determine when the particle changes direction
    (e) Calculate the average velocity over $[0, 5]$

37. The acceleration of an object is $a(t) = 10t - 14$ m/s².
    (a) Find the velocity if $v(0) = 6$ m/s
    (b) Find the position if $s(0) = 4$
    (c) Calculate the displacement from $t = 1$ to $t = 3$
    (d) Find when the object is at rest
    (e) Determine the object's minimum velocity

38. Water flows into a tank at rate $R(t) = 12 + 4t$ liters per minute.
    (a) Find the total volume added in the first 3 minutes
    (b) If the tank initially contains 25 liters, find $V(t)$
    (c) Calculate the average flow rate over 3 minutes
    (d) Find when the tank contains 100 liters
    (e) Determine the rate of acceleration of flow

39. The electric field energy density is $u = \frac{\epsilon E^2}{2}$ where E is field strength.
    (a) Find total energy for uniform field E_0 in volume V
    (b) If $E(x) = E_0 \sin(\frac{\pi x}{d})$, find energy in region $0 \le x \le d$
    (c) Calculate energy if $E_0 = 1000$ V/m, $d = 0.1$ m, area $= 0.01$ m²
    (d) Compare with capacitor energy $U = \frac{1}{2}CV^2$

40. The charge on a capacitor follows $q(t) = Q_0(1 - e^{-t/RC})$ in charging circuit.
    (a) Find current $i(t) = \frac{dq}{dt}$
    (b) Calculate total charge transferred by time $t = 3RC$
    (c) Find the average current over interval $[0, RC]$
    (d) Determine when charge reaches 90% of final value

Section I: Advanced Applications and Techniques

41. The center of mass of a thin rod from $x = a$ to $x = b$ with density $\rho(x)$ is: $\bar{x} = \frac{\int_a^b x\rho(x)\,dx}{\int_a^b \rho(x)\,dx}$
    (a) Find the center of mass of a rod from $x = 0$ to $x = 6$ with density $\rho(x) = 5x + 4$
    (b) Calculate the total mass of the rod
    (c) Find the center of mass if density is $\rho(x) = e^{5x}$
    (d) Compare with uniform density $\rho(x) = 5$

42. The moment of inertia about the x-axis is $I_x = \int y^2\,dm$ where $dm = \rho\,dA$.

(a) Find I_x for the region under $y = 5x^2$ from $x = 0$ to $x = 1$ with uniform density

(b) Calculate the radius of gyration $r_g = \sqrt{\frac{I_x}{M}}$

(c) Find the moment of inertia about the y-axis

(d) Discuss applications in rotational dynamics

43. Arc length of a curve $y = f(x)$ from $x = a$ to $x = b$ is: $L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$

(a) Find the arc length of $y = 5x^2$ from $x = 0$ to $x = 1$

(b) Calculate the arc length of $y = \ln(5x)$ from $x = 1$ to $x = e$

(c) Find the perimeter of one arch of $y = 4\sin x$

(d) Explain the geometric interpretation of the formula

44. Surface area of revolution about x-axis is: $S = 2\pi \int_a^b y\sqrt{1 + (y')^2}\, dx$

(a) Find the surface area when $y = 5x$ from $x = 0$ to $x = 3$ is rotated

(b) Calculate surface area for $y = \sqrt{5x}$ from $x = 0$ to $x = 5$

(c) Find the surface area of a sphere of radius $5R$

(d) Verify using geometric sphere formula $S = 4\pi r^2$

45. Economic applications of integration:

(a) If marginal cost is $MC(x) = 6x + 11$, find total cost function given fixed costs of £300

(b) Calculate consumer surplus if demand is $p = 50 - 5x^2$ and price is £20

(c) Find producer surplus for supply curve $p = 4x^2 + 6$ at equilibrium price £18

(d) Analyze market efficiency at equilibrium

46. Probability density functions satisfy $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

(a) Find the constant g so that $f(x) = gx^6$ is a PDF on $[0, 1]$

(b) Calculate $P(0.4 \leq X \leq 0.9)$ for this distribution

(c) Find the cumulative distribution function $F(x)$

(d) Calculate the median value of the distribution

47. Design an integration problem modeling fluid mechanics:

(a) Define a pressure distribution in a fluid column

(b) Set up integrals for hydrostatic force on surfaces

(c) Calculate buoyancy effects using integration

(d) Interpret results for engineering design

(e) Discuss real-world applications and limitations

48. Error bounds in numerical integration:

(a) Use the trapezoidal rule with $n = 12$ to approximate $\int_0^3 \sqrt{1 + x^3}\, dx$

(b) Apply Simpson's rule with $n = 12$ to the same integral

(c) Estimate theoretical error bounds for each method

(d) Compare computational efficiency of different methods

(e) Research adaptive integration algorithms

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

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