A Level Pure Mathematics
Practice Test 5: Proof

Instructions:
Answer all questions. Show your working clearly.
Calculators may NOT be used in this test.
Time allowed: 2 hours

Section A: Direct Proof

1. Prove that the sum of an even integer and an odd integer is always odd.
2. Prove that if n is an even integer, then $n^2 + n$ is even.
3. Prove that the difference of the squares of two consecutive odd integers is always divisible by 8.
4. Prove that for any integer n, the expression $3n(n + 1)$ is always even.
5. Given that p is rational and q is rational with $q \neq 0$, prove that $p - q$ is rational.
6. Prove that if $a \geq 0$, $b \geq 0$, and $c \geq 0$, then $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$ (AM-GM inequality for three terms).
7. Prove that for any real numbers x and y, $x^2 + y^2 \geq 2xy$ with equality if and only if $x = y$.
8. Prove that in any triangle with sides a, b, and c, if $a \leq b \leq c$, then $c < a + b$.
9. Let $r(x) = x^{11} - 2x^9 + 4x^7 - x^5 + 3x^3 - 5x$. Prove that r is an odd function.
10. Prove that the function $s(x) = -4x + 1$ is strictly decreasing on $\mathbb{R}$.

Section B: Proof by Contradiction

11. Prove that $\sqrt{17}$ is irrational.
12. Prove that there is no largest negative integer.
13. Prove that $\sqrt{12}$ is irrational.
14. Prove that if n^3 is divisible by 7, then n is divisible by 7.
15. Prove that there is no rational number whose cube is 2.
16. Prove that if a and b are integers with $2a^2 + b^2 = 3$, then $a = 0$ and $b = \pm\sqrt{3}$, which is impossible for integer b.
17. Prove that $\log_3 7$ is irrational.
18. Prove that the equation $3x^2 + 2x + 5 = 0$ has no real solutions.
19. Prove that the equation $x^2 - 11y^2 = 6$ has no integer solutions.
20. Prove that if n is an integer and $n^2 + 1$ is even, then n is odd.

Section C: Mathematical Induction - Sequences and Series

21. Prove by induction that $8 + 16 + 24 + \ldots + 8n = 4n(n + 1)$ for all positive integers n.
22. Prove by induction that $3^2 + 6^2 + 9^2 + \ldots + (3n)^2 = \frac{9n(n+1)(2n+1)}{6}$ for all positive integers n.
23. Prove by induction that $6 + 11 + 16 + \ldots + (5n + 1) = \frac{n(5n+7)}{2}$ for all positive integers n.
24. Prove by induction that $9 + 18 + 27 + \ldots + 9n = \frac{9n(n+1)}{2}$ for all positive integers n.
25. Prove by induction that $2 + 7 + 12 + \ldots + (5n - 3) = \frac{n(5n-1)}{2}$ for all positive integers n.
26. Let $t_1 = 4$ and $t_{n+1} = 3t_n - 2$ for $n \geq 1$. Prove by induction that $t_n = 3^n + 1$ for all positive integers n.
27. Prove by induction that $\sum_{r=1}^{n} r \cdot 6^r = \frac{(5n-1)6^{n+1}+6}{25}$ for all positive integers n.
28. Prove by induction that $\sum_{r=1}^{n} \frac{1}{(3r-2)(3r+1)} = \frac{n}{3n+1}$ for all positive integers n.
29. The sequence U_n is defined by $U_1 = 2$, $U_2 = 5$, and $U_{n+1} = U_n + 2U_{n-1}$ for $n \geq 2$. Prove by induction that $U_n = 2^n + (-1)^n$ for all $n \geq 1$.
30. Prove by induction that $\sum_{r=1}^{n} r^2(r + 3) = \frac{n(n+1)(n+2)(3n+11)}{12}$ for all positive integers n.

Section D: Mathematical Induction - Inequalities

31. Prove by induction that $7^n \geq 6n + 1$ for all non-negative integers n.
32. Prove by induction that $5^n \geq 4n^2$ for all integers $n \geq 2$.
33. Prove by induction that $n! \geq 6^{n-5}$ for all integers $n \geq 7$.
34. Prove by induction that $(1 + x)^n \geq 1 + nx + \frac{n(n-1)(n-2)}{6}x^3$ for all real $x \geq 0$ and all integers $n \geq 3$.
35. Prove by induction that $\frac{1}{5^2} + \frac{1}{6^2} + \ldots + \frac{1}{n^2} < \frac{1}{4} - \frac{1}{4n}$ for all integers $n \geq 5$.
36. Prove by induction that $\frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} + \ldots + \frac{1}{\sqrt{n}} \geq 2(\sqrt{n} - 2)$ for all integers $n \geq 5$.
37. Prove by induction that $1 + \frac{1}{5^2} + \frac{1}{6^2} + \ldots + \frac{1}{n^2} < \frac{3}{2}$ for all integers $n \geq 5$.
38. Prove by induction that $6^n \geq 3n^3$ for all integers $n \geq 3$.
39. Prove by induction that $\left(1 + \frac{1}{5n}\right)^n < \frac{4}{3}$ for all positive integers n.
40. Prove by induction that for $n \geq 5$, $\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} \geq \frac{3}{5}$.

Section E: Mathematical Induction - Divisibility

41. Prove by induction that $n^3 + 23n$ is divisible by 6 for all positive integers n.
42. Prove by induction that $8^n - 1$ is divisible by 7 for all positive integers n.
43. Prove by induction that $10^n - 1$ is divisible by 9 for all positive integers n.
44. Prove by induction that $4n^3 + 6n^2 + 2n$ is divisible by 12 for all positive integers n.
45. Prove by induction that $12^n - 1$ is divisible by 11 for all positive integers n.
46. Prove by induction that $6^{2n+1} + 3^{n+2}$ is divisible by 11 for all non-negative integers n.

47. Prove by induction that $14^n - 7^n$ is divisible by 7 for all positive integers n.
48. Prove by induction that $9^{2n} - 1$ is divisible by 80 for all positive integers n.
49. Prove by induction that $n^{17} - n$ is divisible by 17 for all positive integers n.
50. Prove by induction that $17^n + 18^n$ is divisible by 35 for all odd positive integers n.

Section F: Deduction in Algebraic Manipulation

51. Given that $m + n = 9$ and $mn = 20$, find the value of $m^2 + n^2$.
52. If $u + v + w = 3$ and $uv + vw + wu = -2$, find the value of $u^2 + v^2 + w^2$.
53. Given that α and β are roots of $x^2 + x - 3 = 0$, prove that:
    (a) $\alpha + \beta = -1$
    (b) $\alpha\beta = -3$
    (c) $\alpha^3 + \beta^3 = 10$
54. If $w + \frac{1}{w} = 3$, find expressions for:
    (a) $w^2 + \frac{1}{w^2}$
    (b) $w^3 + \frac{1}{w^3}$
    (c) $w^4 + \frac{1}{w^4}$
55. Prove that if $p + q + r = 0$, then $(p + q + r)^2 = p^2 + q^2 + r^2 + 2(pq + qr + rp) = 0$.
56. Given that x, y, z are in geometric progression, prove that x^2, y^2, z^2 are also in geometric progression.
57. If $\cos\alpha + \cos\beta = 1$ and $\sin\alpha + \sin\beta = 1$, prove that $\cos(\alpha - \beta) = \frac{1}{2}$.
58. Prove that $(a - b - c)^2 + (b - c - a)^2 + (c - a - b)^2 = 2(a^2 + b^2 + c^2) - 2ab - 2bc - 2ca$.
59. Given that $\log p$, $\log q$, $\log r$ are in arithmetic progression, prove that $q^2 = pr$.
60. If a, b, c are in harmonic progression, prove that $\frac{a+c}{2} \geq \frac{2ac}{a+c} = b$ (harmonic mean inequality).

Section G: Deduction in Geometric Reasoning

61. In triangle LMN, prove that the sum of any two angles is less than 180 plus the third angle.
62. Prove that if a line bisects two sides of a triangle, then it is parallel to the third side and its length is half that of the third side.
63. Prove that if two chords of a circle are equal, then they are equidistant from the center.
64. In triangle ABC, let H be the orthocenter. Prove that $\angle BHC = 180 - \angle A$ when A is acute.
65. Prove that if both pairs of opposite sides of a quadrilateral are equal, then it is a parallelogram.
66. In a circle, prove that the perpendicular bisector of any chord passes through the center.
67. Prove that if two circles intersect at two points, the line joining their centers is the perpendicular bisector of the common chord.
68. In triangle STU, prove that $\frac{s}{\sin S} = \frac{t}{\sin T} = \frac{u}{\sin U} = 2R$ where R is the circumradius.
69. Prove that the altitudes of a triangle are concurrent (meet at the orthocenter).
70. Prove that in a rhombus, the diagonals bisect each other at right angles.

Section H: Advanced Proof Techniques

71. Prove that the irrational numbers are dense in the real numbers.

72. Prove that if $k(x) = \frac{4x-1}{3x+2}$ where $x \neq -\frac{2}{3}$, then k has an inverse function on its domain.

73. Prove that the set of rational numbers between 0 and 1 is countably infinite.

74. Use the pigeonhole principle to prove that in any group of 10 people, at least two have birthdays in the same month.

75. Prove that $5 + \sqrt{11}$ is irrational.

76. Prove that if p is an odd prime, then 8 divides $p^2 - 1$.

77. Prove that the square root of any non-perfect-square positive integer is irrational.

78. Use strong induction to prove that the Fibonacci sequence F_n satisfies $F_n < 2^n$ for all $n \geq 1$.

79. Prove that if $b_1, b_2, \ldots, b_n$ are positive real numbers, then:

$$\left(\frac{b_1 + b_2 + \ldots + b_n}{n}\right)^2 \geq \frac{b_1^2 + b_2^2 + \ldots + b_n^2}{n}$$

(Cauchy-Schwarz inequality special case)

80. Prove or disprove: For all positive integers n, $6^n - 1$ is divisible by 5.

Section I: Proof Writing and Communication

81. Write a complete proof of Heron's formula: For a triangle with sides a, b, c and semiperimeter $s = \frac{a+b+c}{2}$, the area is $A = \sqrt{s(s-a)(s-b)(s-c)}$.

82. Prove that the equation $x^5 + y^5 = z^2$ has no positive integer solutions when $\gcd(x, y) = 1$.

83. Let $Q_n = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1)$. Prove that $Q_n = \frac{n(n+1)(n+2)}{3}$ and deduce that Q_n is always an integer.

84. Prove Stewart's theorem: In triangle ABC with cevian AD where D lies on BC with $BD = m$ and $DC = n$, we have:

$$b^2 m + c^2 n = a(d^2 + mn)$$

where $a = BC$, $b = AC$, $c = AB$, and $d = AD$.

85. Consider the sequence defined by $g_1 = 1$, $g_2 = 4$, and $g_{n+2} = g_{n+1} + g_n$ for $n \geq 1$. Prove that $\gcd(g_n, g_{n+1}) = 1$ for all $n \geq 1$.

86. Prove that for any positive integer n, the number $9^{2n} - 4^{n+1}$ is divisible by 17.

87. Let $\phi : \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{-3\}$ be defined by $\phi(x) = \frac{-3x+2}{x-1}$. Prove that ϕ is bijective and find ϕ^{-1}.

88. Prove the Fundamental Theorem of Arithmetic: Every integer greater than 1 is either prime or can be expressed uniquely as a product of prime factors (up to order).

89. Prove that $\sqrt[4]{3}$ is irrational by showing that if $\sqrt[4]{3} = \frac{p}{q}$ in lowest terms, then both p and q must be divisible by 3.

90. Write a proof showing that there are infinitely many primes of the form $6k + 5$ where k is a non-negative integer.

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

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