A Level Pure Mathematics
Practice Test 5: Differentiation

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Basic Differentiation - Polynomials

1. Differentiate these polynomial functions:
   (a) $f(x) = 4x^5 - 3x^4 + 2x^3 - 8x + 6$
   (b) $g(x) = 5x^3 + \frac{3}{4}x^2 - 6x + 12$
   (c) $h(x) = (x + 3)(x - 5)$
   (d) $k(x) = (4x - 1)^2$
   (e) $p(x) = x^4(x^2 - 3)$
   (f) $q(x) = \frac{x^6 - 4x^4 + 3x^2}{x^3}$
2. Find $\frac{dy}{dx}$ for:
   (a) $y = 4x^{-1} + 6x^{-3} - 5$
   (b) $y = \frac{5}{x^4} - \frac{3}{\sqrt{x}} + 8\sqrt{x}$
   (c) $y = 3\sqrt{x^7} + \frac{4}{x^5} - x^{-\frac{3}{4}}$
   (d) $y = (3x + \frac{2}{x})^2$
3. Find the gradient of these curves at the given points:
   (a) $y = 2x^3 - 5x^2 + 4x - 3$ at $x = 3$
   (b) $y = x^4 - 3x^3 + 5$ at $x = -1$
   (c) $y = \frac{3x^2+2}{x}$ at $x = 3$
   (d) $y = (x - 3)^3$ at $x = 4$
4. Find the equation of the tangent line to:
   (a) $y = 3x^3 - 2x^2 + 4x - 5$ at the point where $x = 1$
   (b) $y = x^2 - 6x + 8$ at the point $(4, 0)$
   (c) $y = x^3 - 6x$ at the point where the gradient is 12
   (d) $y = \frac{x^2}{4} - x + 3$ at the point where $x = 2$
5. Given that $f(x) = mx^3 + nx^2 + px + q$ and $f'(x) = 12x^2 - 24x + 9$:
   (a) Find the values of m, n, and p
   (b) If $f(0) = 7$, find the value of q
   (c) Write the complete expression for $f(x)$
   (d) Find $f(1)$ and $f'(3)$

Section B: Differentiation of Special Functions

6. Differentiate these exponential and logarithmic functions:
   (a) $f(x) = 5e^x$
   (b) $g(x) = 2e^x + 4x^4$
   (c) $h(x) = x^3e^x$
   (d) $k(x) = 2\ln x$
   (e) $p(x) = x^3 \ln x$
   (f) $q(x) = \frac{\ln x}{x^2}$
7. Differentiate these trigonometric functions:
   (a) $f(x) = 4\sin x - 2\cos x$
   (b) $g(x) = 5\sin x + 3\cos x - x^4$
   (c) $h(x) = x^3 \cos x$
   (d) $k(x) = \frac{\tan x}{x}$
   (e) $p(x) = 3\cot x$
   (f) $q(x) = \csc x$
8. Find $\frac{dy}{dx}$ for:
   (a) $y = e^{4x}$
   (b) $y = \ln(5x)$
   (c) $y = \sin(5x)$
   (d) $y = \cos(4x - 3)$
   (e) $y = e^{3x^2}$
   (f) $y = \ln(x^4 + 3)$
9. Differentiate using appropriate rules:
   (a) $f(x) = e^x \tan x$
   (b) $g(x) = x^4 \sin x$
   (c) $h(x) = \frac{e^x}{x^2}$
   (d) $k(x) = \frac{\tan x}{\cos x}$
   (e) $p(x) = (\ln x)^4$
   (f) $q(x) = \sqrt{\tan x}$
10. Find the derivatives of:
    (a) $f(x) = \tan^2 x$
    (b) $g(x) = \cos^5 x$
    (c) $h(x) = e^{\tan x}$
    (d) $k(x) = \ln(\tan x)$
    (e) $p(x) = (\sin x + \cos x)^3$
    (f) $q(x) = \cos^{-1} x$ (inverse cos)

Section C: Product Rule and Quotient Rule

11. Use the product rule to differentiate:
    (a) $f(x) = (x^4 - 1)(x^2 + 3)$
    (b) $g(x) = (4x + 3)(x^2 - 3x + 2)$
    (c) $h(x) = x^4 e^x$
    (d) $k(x) = (x + 3)\ln x$
    (e) $p(x) = \cos x \tan x$
    (f) $q(x) = x^4 \sin x$
12. Use the quotient rule to differentiate:
    (a) $f(x) = \frac{x^4+2}{x-3}$
    (b) $g(x) = \frac{4x+1}{x^2-1}$
    (c) $h(x) = \frac{e^x}{x^4}$
    (d) $k(x) = \frac{\ln x}{x+3}$
    (e) $p(x) = \frac{\tan x}{1+\cos x}$
    (f) $q(x) = \frac{x^4}{\tan x}$
13. Choose the most appropriate method to differentiate:
    (a) $f(x) = \frac{x^5+4x^3}{x^3}$
    (b) $g(x) = (x^2 + 2)(x^2 - 4)$
    (c) $h(x) = \frac{x^4+2x^2-1}{x^4}$
    (d) $k(x) = x^3(x^3 + 2)^2$
    (e) $p(x) = \frac{(x+2)^4}{x^3}$
    (f) $q(x) = x^4\sqrt{x + 3}$
14. Given $f(x) = x^4$ and $g(x) = \sin x$:
    (a) Find $(fg)'(x)$ using the product rule
    (b) Find $(\frac{f}{g})'(x)$ using the quotient rule
    (c) Evaluate $(fg)'(\frac{\pi}{6})$
    (d) Evaluate $(\frac{f}{g})'(\frac{\pi}{2})$
15. Prove these differentiation rules:
    (a) Product rule: $(uv)' = u'v + uv'$
    (b) Quotient rule: $(\frac{u}{v})' = \frac{u'v-uv'}{v^2}$
    (c) Show that $(\frac{1}{v})' = -\frac{v'}{v^2}$
    (d) Verify that $(uvw)' = u'vw + uv'w + uvw'$

Section D: Chain Rule

16. Use the chain rule to differentiate:
    (a) $f(x) = (4x + 3)^5$
    (b) $g(x) = (x^2 + 4x - 2)^6$
    (c) $h(x) = \sqrt{3x^2 + 1}$
    (d) $k(x) = (5x - 3)^{-4}$
    (e) $p(x) = \sin(4x + 2)$
    (f) $q(x) = \cos(x^4)$

17. Find $\frac{dy}{dx}$ for:
    (a) $y = e^{4x-2}$
    (b) $y = \ln(3x + 7)$
    (c) $y = (x^2 + 4x)^7$
    (d) $y = \tan^2 x$
    (e) $y = \cos(e^x)$
    (f) $y = e^{\tan x}$

18. Differentiate these composite functions:
    (a) $f(x) = (e^x + 2)^5$
    (b) $g(x) = \ln(x^2 + 4x + 1)$
    (c) $h(x) = \sin(\ln x)$
    (d) $k(x) = e^{x \tan x}$
    (e) $p(x) = (\sin x - \cos x)^4$
    (f) $q(x) = \ln(\tan x)$

19. Use multiple rules to differentiate:
    (a) $f(x) = x^3(4x + 1)^5$
    (b) $g(x) = \frac{x^4}{(x+2)^4}$
    (c) $h(x) = x^4 \sin(4x)$
    (d) $k(x) = e^x \cos(4x)$
    (e) $p(x) = \frac{\ln x}{\sqrt{x^3+2}}$
    (f) $q(x) = \frac{(x^2+3)^4}{x^3}$

20. Find the second derivatives:
    (a) $f(x) = (x + 3)^6$
    (b) $g(x) = \sin(4x)$
    (c) $h(x) = e^{3x}$
    (d) $k(x) = \ln(x^4)$
    (e) $p(x) = x^4 e^x$
    (f) $q(x) = \cos x \cos x$

Section E: Stationary Points

21. Find the coordinates of stationary points for:
    (a) $f(x) = x^3 - 9x^2 + 24x - 5$
    (b) $g(x) = 4x^3 - 12x^2 + 9x + 3$
    (c) $h(x) = x^4 - 12x^2 + 20$
    (d) $k(x) = \frac{x^2}{x-2}$ for $x \neq 2$
22. Determine the nature of each stationary point using the second derivative test:
    (a) $f(x) = x^3 - 12x^2 + 36x + 8$
    (b) $g(x) = 4x^3 - 9x^2 - 30x + 1$
    (c) $h(x) = x^4 - 6x^2 + 9$
    (d) $k(x) = x^3e^{-x}$
23. Find and classify all stationary points:
    (a) $f(x) = x^3 - 6x^2 + 9x - 2$
    (b) $g(x) = 4x^3 - 5x^2 - 24x + 3$
    (c) $h(x) = x^4 - 16x^2 + 64$
    (d) $k(x) = 2x + \frac{8}{x}$ for $x > 0$
24. For the function $f(x) = cx^3 + dx^2 + ex + f$:
    (a) Find the conditions on c, d, and e for the function to have two stationary points
    (b) If $f(x) = 3x^3 - 9x^2 + 9x + 2$, show it has no stationary points
    (c) Find the values of h for which $f(x) = x^3 - 9hx + 4$ has exactly one stationary point
25. Analyze the function $f(x) = \frac{x^2-16}{x}$:
    (a) Find the domain of $f(x)$
    (b) Find $f'(x)$ and locate stationary points
    (c) Determine the nature of stationary points
    (d) Find any asymptotes
    (e) Sketch the graph of $y = f(x)$

Section F: Rates of Change

26. A particle moves along a line with position $s(t) = 3t^3 - 12t^2 + 15t + 8$ meters at time t seconds.
    (a) Find the velocity $v(t)$ and acceleration $a(t)$
    (b) Find when the particle is at rest
    (c) Calculate the velocity and acceleration at $t = 4$
    (d) Determine when the acceleration is zero
    (e) Find the displacement between $t = 1$ and $t = 5$
27. The surface area of a sphere is $A = 4\pi r^2$. If the radius increases at a rate of 4 cm/s:
    (a) Find the rate of change of surface area when $r = 3$ cm
    (b) Express $\frac{dA}{dt}$ in terms of r and $\frac{dr}{dt}$
    (c) When is the surface area increasing at 200π cm²/s?

(d) Find the rate of change of volume when $r = 6$ cm

28. A ladder 10 meters long leans against a vertical wall. The bottom slides away at 2.5 m/s.

    (a) Set up the relationship between distances
    (b) Find how fast the top slides down when the bottom is 8m from the wall
    (c) Find the rate of change of the angle with the ground
    (d) When is the top sliding down fastest?

29. Water flows into a conical tank (vertex up) at 4 m^3/min. The tank has height 8m and radius 4m.

    (a) Express the volume in terms of height h
    (b) Find how fast the water level rises when $h = 3$m
    (c) Find the rate of change of radius when $h = 6$m
    (d) When is the water level rising fastest?

30. The value of an investment follows $V(t) = 8000e^{0.04t}$ where t is years.

    (a) Find the growth rate $\frac{dV}{dt}$
    (b) Calculate the value and growth rate after 3 years
    (c) When is the investment growing at £400 per year?
    (d) Express the growth rate as a percentage of current value

Section G: Optimization Problems

31. A farmer has 400m of fencing to enclose a rectangular field with two dividers parallel to one side.

    (a) Express the area in terms of one variable
    (b) Find the dimensions for maximum area
    (c) Calculate the maximum area
    (d) Verify this is a maximum using the second derivative

32. A closed rectangular box with square base has volume 64 m^3. The material for the base costs £6/m^2, sides cost £4/m^2, and top costs £3/m^2.

    (a) Express the cost in terms of the base side length
    (b) Find dimensions for minimum cost
    (c) Calculate the minimum cost
    (d) Find the ratio of height to base side length

33. A company's cost function is $C(x) = x^3 - 18x^2 + 96x + 200$ thousand pounds, where x is production level (thousands of units).

    (a) Find the production levels for maximum and minimum cost
    (b) Calculate the minimum cost
    (c) Find the marginal cost function
    (d) Determine the optimal production level

34. A rectangular garden is enclosed by a fence and divided by a path. The total fencing (including path) is 100m.

(a) Express the garden area in terms of width
(b) Find dimensions for maximum garden area
(c) Calculate the maximum garden area
(d) Find the ratio of garden length to width

35. A right circular cone has fixed slant height of 15 cm. Find dimensions to maximize volume.
    (a) Express volume in terms of base radius
    (b) Find the critical points
    (c) Determine optimal radius and height
    (d) Calculate maximum volume
    (e) Verify this gives a maximum

Section H: Implicit Differentiation and Related Rates

36. Find $\frac{dy}{dx}$ using implicit differentiation:
    (a) $x^2 + y^2 = 49$
    (b) $x^2 + 4xy + y^2 = 20$
    (c) $x^3 + y^3 = 12xy$
    (d) $\tan(xy) = x + y$
    (e) $e^{xy} = 2x + y$
    (f) $\ln(x^2 + y^2) = xy$

37. Find the equation of the tangent to these curves at the given points:
    (a) $x^2 + y^2 = 20$ at $(2, 4)$
    (b) $x^2 + xy + y^2 = 21$ at $(3, 2)$
    (c) $x^3 + y^3 = 16$ at $(2, 2)$
    (d) $xe^y = 6$ at $(3, \ln 2)$

38. Use implicit differentiation to find $\frac{d^2y}{dx^2}$:
    (a) $x^2 + y^2 = 25$
    (b) $xy = 9$
    (c) $x^2 - y^2 = 16$

39. Two planes start from airports 80 km apart. Plane A travels north at 500 km/h, Plane B travels east at 600 km/h.
    (a) Express the distance between planes as a function of time
    (b) Find how fast they're separating after 0.5 hours
    (c) When are they separating at 900 km/h?
    (d) Find the minimum distance between them

40. A spherical soap bubble is expanding so its volume increases at 120 cm³/s. Find the rate of increase of:
    (a) Radius when $r = 4$ cm
    (b) Surface area when $r = 8$ cm
    (c) Diameter when volume is 4000 cm³
    (d) The rate when surface area is 500π cm²

Section I: Advanced Applications

41. A Moorish arch has the shape of a rectangle topped by a semicircle, with total height 16m.
    (a) Find dimensions to maximize the area
    (b) Calculate the maximum area
    (c) Find the optimal ratio of rectangle height to width
    (d) Determine what fraction of area is rectangular
42. The load capacity of a rectangular beam is proportional to wd^2 where w is width and d is depth. A beam is cut from a cylindrical log of radius 18 cm.
    (a) Express load capacity in terms of width w
    (b) Find dimensions for maximum load capacity
    (c) Calculate the ratio $\frac{d}{w}$ for strongest beam
    (d) Compare with beam of square cross-section
43. A drug absorption rate follows $R(t) = \frac{Ct^3}{(t+3)^4}$ mg/h where t is hours after administration.
    (a) Find when absorption rate is maximum
    (b) If peak rate is 5 mg/h, find C
    (c) Calculate the rate of change at $t = 3$
    (d) Find when absorption rate is decreasing fastest
    (e) Determine the time for half-peak absorption rate
44. A rectangle is inscribed in an ellipse with equation $\frac{x^2}{16} + \frac{y^2}{9} = 1$.
    (a) Express the rectangle area in terms of x-coordinate
    (b) Find coordinates for maximum area
    (c) Calculate the maximum area
    (d) Show the optimal rectangle has specific proportions
45. A delivery service's daily profit is $P(n) = 0.3n^3 - 3.6n^2 + 12n + 80$ for making n hundred deliveries.
    (a) Find the marginal profit function
    (b) Determine the number of deliveries for maximum profit
    (c) Calculate the maximum daily profit
    (d) Find when marginal profit equals average profit
    (e) Graph the profit function and interpret economically
46. Two positive numbers have sum 36. Find the numbers that:
    (a) Maximize their product
    (b) Minimize the sum of their cubes
    (c) Maximize the product of their square roots
    (d) Minimize $x^4 + y^2$ where $x + y = 36$
    (e) Explain why the answers differ
47. A water rocket follows the trajectory $y = x \tan\beta - \frac{gx^2}{2v_0^2 \cos^2\beta}$ where β is launch angle.
    (a) Find the range (horizontal distance when $y = 0$)

(b) Find the angle for maximum range
(c) Calculate the maximum height achieved
(d) Find the angle for maximum height at distance x
(e) Derive the optimal trajectory envelope

48. Create a real-world optimization scenario from engineering:

(a) Define your problem and variables clearly
(b) Set up the objective function and constraints
(c) Use differentiation to find optimal solutions
(d) Verify your solution makes engineering sense
(e) Discuss practical limitations of your model

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

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