A Level Pure Mathematics
Practice Test 3: Sequences and Series

Instructions:
Answer all questions. Show your working clearly.
Calculators may be used unless stated otherwise.
Time allowed: 2 hours

Section A: Arithmetic Sequences

1. For the arithmetic sequence $9, 15, 21, 27, 33, \ldots$:
   (a) Find the first term a and common difference d
   (b) Find the general term u_n
   (c) Calculate u_{30}
   (d) Find which term equals 129
   (e) Determine if 200 is a term in the sequence

2. An arithmetic sequence has $u_5 = 31$ and $u_{11} = 55$.
   (a) Find the first term and common difference
   (b) Write the general term u_n
   (c) Calculate u_{18}
   (d) Find the first term to exceed 200
   (e) Determine the largest value of n for which $u_n < 250$

3. The nth term of an arithmetic sequence is $u_n = 5n - 3$.
   (a) Write down the first five terms
   (b) Find the common difference
   (c) Calculate u_{35}
   (d) Find the sum of the first 25 terms
   (e) For what value of n is $u_n = 147$?

4. Three numbers $p - 3k$, p, and $p + 3k$ are in arithmetic progression with sum 45 and product 2835.
   (a) Find the value of p
   (b) Set up an equation for k
   (c) Solve to find the values of k
   (d) Write down the three numbers for each case

5. An arithmetic sequence has first term a and common difference d.

(a) If the sum of the first p terms equals the sum of the first q terms (where $p \neq q$), prove that the sum of the first $(p+q)$ terms is zero

(b) Show that for any arithmetic sequence, $u_r + u_s = u_t + u_u$ when $r + s = t + u$

(c) Prove that the sum of an arithmetic sequence is $S_n = \frac{n}{2}[2a + (n-1)d]$

(d) If three terms u_i, u_j, u_k are in geometric progression, show that i, j, k cannot be in arithmetic progression unless the sequence is constant

Section B: Arithmetic Series

6. Calculate the sum of these arithmetic series:

   (a) $6 + 11 + 16 + 21 + \ldots$ (first 22 terms)

   (b) $30 + 26 + 22 + 18 + \ldots$ (first 15 terms)

   (c) $\frac{1}{3} + \frac{2}{3} + 1 + \frac{4}{3} + \ldots$ (first 30 terms)

   (d) The series with first term 15, last term 105, and 16 terms

7. An arithmetic series has first term 9 and common difference 5.

   (a) Find the sum of the first 18 terms

   (b) Find the smallest value of n for which $S_n \geq 1500$

   (c) If the sum of the first n terms is 1224, find n

   (d) Express S_n in terms of n

8. The sum of the first n terms of an arithmetic series is $S_n = n^2 + 4n$.

   (a) Find the first term u_1

   (b) Find u_2 and u_3

   (c) Determine the common difference

   (d) Find the general term u_n

   (e) Verify using the formula $u_n = S_n - S_{n-1}$ for $n \geq 2$

9. Find the sum of:

   (a) All multiples of 6 between 200 and 800

   (b) All integers from 1 to 120 that are divisible by 7

   (c) All even integers from 10 to 200

   (d) The integers from 1 to 100 that are divisible by 2 or 3

10. An arithmetic series has $S_{12} = 240$ and $S_{18} = 522$.

    (a) Find the first term and common difference

    (b) Calculate S_{25}

    (c) Find the 10th term

    (d) Determine when the sum first exceeds 1000

Section C: Geometric Sequences

11. For the geometric sequence $4, 12, 36, 108, 324, \ldots$:
    (a) Find the first term a and common ratio r
    (b) Find the general term u_n
    (c) Calculate u_9
    (d) Find which term equals 972
    (e) Determine if 2916 is a term in the sequence

12. A geometric sequence has $u_4 = 24$ and $u_7 = 192$.
    (a) Find the common ratio r
    (b) Find the first term a
    (c) Write the general term u_n
    (d) Calculate u_{12}
    (e) Find the first term to exceed 100000

13. The nth term of a geometric sequence is $u_n = 6 \times 3^{n-1}$.
    (a) Write down the first five terms
    (b) Find the common ratio
    (c) Calculate u_{10}
    (d) Find the sum of the first 7 terms
    (e) For what value of n is $u_n = 4374$?

14. Three numbers $\frac{w}{t}$, w, and wt are in geometric progression with sum 91 and product 729.
    (a) Find the value of w
    (b) Set up an equation for t
    (c) Solve to find the values of t
    (d) Write down the three numbers for each case

15. A geometric sequence has first term a and common ratio r.
    (a) If $u_i \cdot u_j = u_k \cdot u_l$, prove that $i + j = k + l$
    (b) Show that the sequence of reciprocals $\frac{1}{u_1}, \frac{1}{u_2}, \frac{1}{u_3}, \ldots$ is also geometric
    (c) Prove that if u_p, u_q, u_r are three terms of a geometric sequence, then $u_p \cdot u_r = u_q^2$ when p, q, r are in arithmetic progression
    (d) Show that $\log u_1, \log u_2, \log u_3, \ldots$ form an arithmetic sequence (when $a > 0$ and $r > 0$)

Section D: Geometric Series

16. Calculate the sum of these geometric series:
    (a) $7 + 21 + 63 + 189 + \ldots$ (first 9 terms)
    (b) $2 - 6 + 18 - 54 + \ldots$ (first 12 terms)
    (c) $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \ldots$ (first 15 terms)
    (d) $64 + 48 + 36 + 27 + \ldots$ (first 10 terms)

17. A geometric series has first term 12 and common ratio $\frac{2}{3}$.

(a) Find the sum of the first 15 terms
(b) Find the smallest value of n for which $S_n \geq 35$
(c) Calculate the sum to infinity
(d) Find how many terms are needed for the sum to be within 0.05 of the sum to infinity

18. The sum of the first n terms of a geometric series is $S_n = 5(4^n - 1)$.
    (a) Find the first term u_1
    (b) Find u_2 and u_3
    (c) Determine the common ratio
    (d) Find the general term u_n
    (e) Verify using the formula $u_n = S_n - S_{n-1}$ for $n \geq 2$

19. Evaluate these infinite geometric series:
    (a) $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots$
    (b) $6 - 3 + \frac{3}{2} - \frac{3}{4} + \ldots$
    (c) $\frac{4}{5} + \frac{4}{25} + \frac{4}{125} + \frac{4}{625} + \ldots$
    (d) $0.6 + 0.06 + 0.006 + 0.0006 + \ldots$

20. A geometric series has $S_5 = 62$ and $S_{10} = 1922$.
    (a) Set up equations for the first term and common ratio
    (b) Solve to find a and r
    (c) Calculate S_{15}
    (d) Find the sum to infinity (if it exists)
    (e) Determine the first term to exceed 5000

Section E: Sigma Notation

21. Evaluate these sums:
    (a) $\sum_{r=1}^{15}(4r + 2)$
    (b) $\sum_{r=1}^{30}(5r - 4)$
    (c) $\sum_{r=1}^{22} r^2$
    (d) $\sum_{r=1}^{14}(3r^2 + r)$

22. Express these series using sigma notation:
    (a) $8 + 13 + 18 + 23 + \ldots + 48$
    (b) $4 + 20 + 100 + 500 + \ldots + 62500$
    (c) $1^3 + 3^3 + 5^3 + 7^3 + \ldots + 19^3$
    (d) $\frac{1}{4} + \frac{1}{12} + \frac{1}{24} + \frac{1}{40} + \ldots + \frac{1}{132}$

23. Use the standard formulae to evaluate:
    (a) $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$: Find $\sum_{r=1}^{60} r$
    (b) $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$: Find $\sum_{r=1}^{30} r^2$
    (c) $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$: Find $\sum_{r=1}^{18} r^3$
    (d) $\sum_{r=1}^{35}(4r^2 - 3r + 2)$

24. Simplify these expressions:
    (a) $\sum_{r=1}^{n}(pr+q)$ in terms of p, q, and n
    (b) $\sum_{r=1}^{n}(3r^2-r+2)$
    (c) $\sum_{r=1}^{n}(2r+3)^2$
    (d) $\sum_{r=1}^{n} r(3r+2)$
25. Prove these results:
    (a) $\sum_{r=1}^{n}(4r-3)=n(2n+1)$
    (b) $\sum_{r=1}^{n} r(r+3)=\frac{n(n+1)(n+8)}{3}$
    (c) $\sum_{r=1}^{n} \frac{1}{(3r-2)(3r+1)}=\frac{n}{3n+1}$
    (d) $\sum_{r=1}^{n}((r+1)^2-r^2)=n(n+2)$

Section F: Binomial Expansion - Integer Powers

26. Expand using the binomial theorem:
    (a) $(x+4)^4$
    (b) $(4x-3)^5$
    (c) $(3-2x)^6$
    (d) $(3x+\frac{2}{x})^4$
27. Find the specified terms in these expansions:
    (a) The coefficient of x^5 in $(4x+2)^9$
    (b) The coefficient of x^7 in $(2x-1)^{10}$
    (c) The constant term in $(x^4+\frac{3}{x^2})^6$
    (d) The coefficient of x^{-1} in $(4x^2-\frac{1}{x})^8$
28. Use the binomial theorem to evaluate:
    (a) $(1.04)^5$ to 6 decimal places
    (b) $(0.95)^4$ to 5 decimal places
    (c) $(1.01)^6$ exactly
    (d) 97^3 by writing it as $(100-3)^3$
29. In the expansion of $(1+cx)^p$:
    (a) The coefficient of x is 18 and the coefficient of x^2 is 135. Find c and p.
    (b) Find the coefficient of x^3
    (c) Write out the first four terms of the expansion
    (d) For what values of x does the expansion converge?
30. The coefficient of x^j in the expansion of $(1+x)^p$ is $\binom{p}{j}$.
    (a) Show that $\binom{p}{0}\cdot 2^0+\binom{p}{1}\cdot 2^1+\binom{p}{2}\cdot 2^2+\ldots+\binom{p}{p}\cdot 2^p=3^p$
    (b) Prove that $\binom{p}{j}\cdot\binom{j}{k}=\binom{p}{k}\cdot\binom{p-k}{j-k}$ for $k\le j\le p$
    (c) Use the identity $\binom{p}{j}=\binom{p-1}{j-1}+\binom{p-1}{j}$ to compute $\binom{10}{4}$
    (d) Show that $\sum_{j=0}^{p}(-1)^j\binom{p}{j}j=0$ for $p\ge 1$

Section G: Binomial Expansion - Non-Integer Powers

31. Expand these expressions up to and including the term in x^3:
    (a) $(1+x)^{2/3}$
    (b) $(1-x)^{-3}$
    (c) $(1+4x)^{-1/2}$
    (d) $(1-5x)^{1/4}$

32. Find the first four terms in the expansion of:
    (a) $(25+x)^{1/2}$
    (b) $(4-x)^{-1/2}$
    (c) $\frac{1}{(3+x)^3}$
    (d) $\sqrt{9-2x}$

33. State the range of values of x for which these expansions are valid:
    (a) $(1+5x)^{-1} = 1 - 5x + 25x^2 - 125x^3 + \dots$
    (b) $(1-4x)^{1/2} = 1 - 2x - 2x^2 - 4x^3 - \dots$
    (c) $(5+x)^{-1} = \frac{1}{5} - \frac{x}{25} + \frac{x^2}{125} - \frac{x^3}{625} + \dots$
    (d) $\frac{1}{\sqrt{16-x}} = \frac{1}{4} + \frac{x}{128} + \frac{3x^2}{8192} + \dots$

34. Use binomial expansions to find approximations:
    (a) $\sqrt{1.06}$ to 5 decimal places
    (b) $\frac{1}{\sqrt{0.92}}$ to 4 decimal places
    (c) $(1.04)^{-4}$ to 6 decimal places
    (d) $\sqrt[3]{1.09}$ to 5 decimal places

35. Find the coefficient of x^2 in the expansion of:
    (a) $(1+x)^{2/3}(1-x)^{1/3}$
    (b) $(1+4x)^{-1}(1+2x)^2$
    (c) $\frac{1+2x}{\sqrt{1-x}}$
    (d) $(1+x-x^2)(1+x)^{-3}$

Section H: Mixed Series and Advanced Topics

36. A sequence is defined by $u_1 = 4$ and $u_{n+1} = 3u_n - 7$ for $n \geq 1$.
    (a) Find the first five terms
    (b) Prove by induction that $u_n = \frac{1}{2}(3^n + 7)$
    (c) Calculate u_{15}
    (d) Find the sum of the first 12 terms

37. The sequence $\{z_n\}$ satisfies $z_n = 4z_{n-1} - 3z_{n-2}$ with $z_1 = 2$ and $z_2 = 5$.
    (a) Find the first six terms
    (b) Show that the characteristic equation is $r^2 - 4r + 3 = 0$
    (c) Solve to find $r = 3$ and $r = 1$

(d) Use the general solution $z_n = A \cdot 3^n + B \cdot 1^n$ to find A and B

(e) Write the explicit formula for z_n

38. Consider the series $\sum_{r=1}^{\infty} \frac{1}{r(r+3)}$.

(a) Use partial fractions to show that $\frac{1}{r(r+3)} = \frac{1}{3}\left(\frac{1}{r} - \frac{1}{r+3}\right)$

(b) Write out the first few terms and observe the telescoping pattern

(c) Find the sum of the first n terms

(d) Determine the sum to infinity

39. The Tribonacci sequence is defined by $T_1 = 1$, $T_2 = 1$, $T_3 = 2$, and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 4$.

(a) Write down the first 12 terms

(b) Calculate the ratios $\frac{T_{n+1}}{T_n}$ for $n = 1, 2, 3, \ldots, 11$

(c) Show that these ratios approach approximately 1.839

(d) Investigate the characteristic equation $x^3 - x^2 - x - 1 = 0$ and its dominant root

40. A spring oscillates with decreasing amplitude. Each oscillation has amplitude $\frac{7}{8}$ of the previous one. The first oscillation has amplitude 8 cm.

(a) Find the amplitude of the 12th oscillation

(b) Calculate the total distance traveled by the spring tip when it comes to rest

(c) Find the number of oscillations needed to reduce the amplitude to less than 1 cm

(d) Model the energy loss (proportional to amplitude squared) and find the total energy dissipated

Section I: Applications and Problem Solving

41. A car loan of £25,000 is taken out at 9% annual compound interest. Monthly payments of £450 are made.

(a) Set up a recurrence relation for the amount owed after n months

(b) Find the amount owed after 18 months

(c) Determine how many months it takes to pay off the loan

(d) Calculate the total amount of interest paid

42. A virus spreads through a population. Each infected person infects 3 others every day. Initially, 5 people are infected.

(a) Model the number of newly infected people each day as a geometric sequence

(b) Find the number of newly infected people on day 8

(c) After how many days will the daily new infections exceed 50,000?

(d) If intervention reduces the infection rate to 1.5 per person per day after day 6, find the total infected after 12 days

43. A snowflake fractal is constructed where each iteration adds smaller triangles with areas forming the sequence: 27, 9, 3, 1, $\frac{1}{3}$, ... cm^2.

(a) Find the total area of all the triangular additions

(b) If the perimeter of each triangle is proportional to the square root of its area with proportionality constant 2, find the total added perimeter

(c) If etching costs £3 per cm of perimeter, find the total etching cost

(d) What percentage of the total area is contributed by the first 3 iterations?

44. A water tank has a leak. Initially, it contains 500 liters. Every hour, 8% leaks out, and 20 liters is added from a refill system.

    (a) Set up a recurrence relation for the volume after n hours

    (b) Find the volume after 12 hours

    (c) Determine the long-term equilibrium volume

    (d) After how many hours is the volume within 5% of the equilibrium?

45. An annuity plan involves depositing £3000 in the first year, £3300 in the second year, £3630 in the third year, and so on (increasing by 10% each year) for 25 years.

    (a) Model the annual deposits as a geometric sequence

    (b) Find the total amount deposited over 25 years

    (c) If each deposit earns 6% annual compound interest from when it's made, find the total value after 25 years

    (d) Compare this with depositing £3000 annually at 6% compound interest for 25 years

Answer Space

Use this space for your working and answers.

END OF TEST

Total marks: 150

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